Module 4 : Lecture 1
COMPRESSIBLE FLOWS
(Fundamental Aspects: Part - I)

Overview

In general, the liquids and gases are the states of a matter that comes under the same category as “fluids”. The incompressible flows are mainly deals with the cases of constant density. Also, when the variation of density in the flow domain is negligible, then the flow can be treated as incompressible. Invariably, it is true for liquids because the density of liquid decreases slightly with temperature and moderately with pressure over a broad range of operating conditions. Hence, the liquids are considered as incompressible. On the contrary, the compressible flows are routinely defined as “variable density flows”. Thus, it is applicable only for gases where they may be considered as incompressible/compressible, depending on the conditions of operation. During the flow of gases under certain conditions, the density changes are so small that the assumption of constant density can be made with reasonable accuracy and in few other cases the density changes of the gases are very much significant (e.g. high speed flows). Due to the dual nature of gases, they need special attention and the broad area of in the study of motion of compressible flows is dealt separately in the subject of “gas dynamics”. Many engineering tasks require the compressible flow applications typically in the design of a building/tower to withstand winds, high speed flow of air over cars/trains/airplanes etc. Thus, gas dynamics is the study of fluid flows where the compressibility and the temperature changes become important. Here, the entire flow field is dominated by Mach waves and shock waves when the flow speed becomes supersonic. Most of the flow properties change across these waves from one state to other. In addition to the basic fluid dynamics, the knowledge of thermodynamics and chemical kinetics is also essential to the study of gas dynamics.

Thermodynamic Aspects of Gases

In high speed flows, the kinetic energy per unit mass $\left(V^2/2\right)$ is very large which is substantial enough to strongly interact with the other properties of the flow. Since the science of energy and entropy is the thermodynamics, it is essential to study the thermodynamic aspects of gases under the conditions compressible high speed flows.

Perfect gas: A gas is considered as a collection of particles (molecules, atoms, ions, electrons etc.) that are in random motion under certain intermolecular forces. These forces vary with distances and thus influence the microscopic behavior of the gases. However, the thermodynamic aspect mainly deals with the global nature of the gases. Over wide ranges of pressures and temperatures in the compressible flow fields, it is seen that the average distance between the molecules is more than the molecular diameters (about 10-times). So, all the flow properties may be treated as macroscopic in nature. A perfect gas follows the relation of pressure, density and temperature in the form of the fundamental equation.

$$p = \rho RT; \;\; R = \frac{\bar{R}}{M} \tag{4.1.1}$$

Here, M is the molecular weight of the gas, R is the gas constant that varies from gas to gas and $\bar{R}\left(=8314\,\mathrm{J/kg.K}\right)$ is the universal gas constant. In a calorically perfect gas, the other important thermodynamic properties relations are written as follows;

$$\begin{aligned} & c_p = \left(\frac{\partial h}{\partial T}\right)_p; \;\; c_v = \left(\frac{\partial e}{\partial T}\right)_v; \; c_p - c_v = R \\ & c_p = \frac{\gamma R}{\gamma - 1}; \; c_v = \frac{R}{\gamma - 1}; \; \gamma = \frac{c_p}{c_v} \end{aligned} \tag{4.1.2}$$

In Eq. (4.1.2), the parameters are specific heat at constant pressure $\left(c_p\right)$, specific heat at constant volume $\left(c_v\right)$, specific heat ratio$\left(\gamma\right)$, specific enthalpy $\left(h\right)$ and specific internal energy $\left(e\right)$.

First law of thermodynamics: A system is a fixed mass of gas separated from the surroundings by a flexible boundary. The heat added (q) and work done (w) on the system can cause change in energy. Since, the system is stationary, the change in internal energy. By definition of first law, we write,

$$\delta q + \delta w = de \tag{4.1.3}$$

For a given de, there are infinite number of different ways by which heat can be added and work done on the system. Primarily, the three common types of processes are, adiabatic (no addition of heat), reversible (no dissipative phenomena) and isentropic (i.e. reversible and adiabatic).

Second law of thermodynamics: In order to ascertain the direction of a thermodynamic process, a new state variable is defined as 'entropy (s)'. The change in entropy during any incremental process (ds) is equal to the actual heat added divided by the temperature (dq/T), plus a contribution from the irreversible dissipative phenomena (ds_{irrev}) occurring within the system.

$$ds = \frac{\delta q}{T} + ds_{irrev} \tag{4.1.4}$$

Since, the dissipative phenomena always increases the entropy, it follows that

$$ds \geq \frac{\delta q}{T};\ \ ds \geq 0\ (\text{Adiabatic process}) \tag{4.1.5}$$

Eqs. (4.1.4 & 4.1.5) are the different forms of *second law of thermodynamics*. In order to calculate the change in entropy of a thermodynamic process, two fundamental relations are used for a calorically perfect gas by combining both the laws of thermodynamics;

$$\begin{aligned} s_2 - s_1 &= c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \\ s_2 - s_1 &= c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\rho_1}{\rho_2}\right) \end{aligned} \tag{4.1.6}$$

An isentropic process is the one for which the entropy is constant and the process is reversible and adiabatic. The isentropic relation is given by the following relation;

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{4.1.7}$$

Important Properties of Compressible Flows

The simple definition of compressible flow is the variable density flows. In general, the density of gases can vary either by changes in pressure and temperature. In fact, all the high speed flows are associated with significant pressure changes. So, let us recall the following fluid properties important for compressible flows;

<u>Bulk modulus</u> (E_v): It is the property of that fluid that represents the variation of density (ρ) with pressure (p) at constant temperature (T). Mathematically, it is represented as,

$$E_v = -\cancel{v}\left(\frac{\partial p}{\partial \cancel{v}}\right)_T = \rho\left(\frac{\partial \rho}{\partial T}\right)_T \tag{4.1.8}$$

In terms of finite changes, it is approximated as,

$$E_v = \frac{(\Delta \cancel{v}/\cancel{v})}{\Delta T} = -\frac{(\Delta \rho/\rho)}{\Delta T} \tag{4.1.9}$$

<u>Coefficient of volume expansion</u> (β): It is the property of that fluid that represents the variation of density with temperature at constant pressure. Mathematically, it is represented as,

$$\beta = \frac{1}{\cancel{v}}\left(\frac{\partial \cancel{v}}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{4.1.10}$$

In terms of finite changes, it is approximated as,

$$\beta = \frac{(\Delta \cancel{v}/\cancel{v})}{\Delta T} = -\frac{(\Delta \rho/\rho)}{\Delta T} \tag{4.1.11}$$

Compressibility(κ): It is defined as the fractional change in the density of the fluid element per unit change in pressure. One can write the expression for κ as follows;

$$\kappa = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right) \quad \Rightarrow d\rho = \rho\kappa\, dp \tag{4.1.12}$$

In order to be more precise, the compression process for a gas involves increase in temperature depending on the amount of heat added or taken away from the gas. If the temperature of the gas remains constant, the definition is refined as *isothermal compressibility*(κ_T). On the other hand, when no heat is added/taken away from the gases and in the absence of any dissipative mechanisms, the compression takes place isentropically. It is then, called as *isentropic compressibility*(κ_s).

$$\kappa_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T; \quad \kappa_s = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s \tag{4.1.13}$$

Being the property of a fluid, the gases have high values of compressibility $\left(\kappa_T = 10^{-5}\,\mathrm{m^2/N} \text{ for air at 1atm}\right)$ while liquids have low values of compressibility much less than that of gases $\left(\kappa_T = 5\times10^{-10}\,\mathrm{m^2/N} \text{ for water at 1atm}\right)$. From the basic definition (Eq. 4.1.12), it is seen that whenever a fluid experiences a change in pressure dp, there will be a corresponding change in $d\rho$. Normally, high speed flows involve large pressure gradient. For a given change in dp, the resulting change in density will be small for liquids (low values of κ) and more for gases (high values of κ). Therefore, for the flow of liquids, the relative large pressure gradients can create much high velocities without much change in densities. Thus, the liquids are treated to be incompressible. On the other hand, for the flow of gases, the moderate to strong pressure gradient leads to substantial changes in the density (Eq.4.1.12) and at the same time, it can create large velocity changes. Such flows are defined as compressible flows where the density is a variable property and the fractional change in density $(d\rho/\rho)$ is too large to be ignored.

Fundamental Equations for Compressible Flow

Consider a compressible flow passing through a rectangular control volume as shown in Fig. 4.1.1. The flow is one-dimensional and the properties change as a function of *x*, from the region ‘1’ to ‘2’ and they are velocity(u), pressure(p), temperature(T), density (ρ) and internal energy(e). The following assumptions are made to derive the fundamental equations;

- Flow is uniform over left and right side of control volume.
- Both sides have equal area (A), perpendicular to the flow.
- Flow is inviscid, steady and nobody forces are present.
- No heat and work interaction takes place to/from the control volume.

Let us apply mass, momentum and energy equations for the one dimensional flow as shown in Fig. 4.1.1.

Conservation of Mass:

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0 \quad \Rightarrow \rho_1 u_1 = \rho_2 u_2 \tag{4.1.14}$$

Conservation of Momentum:

$$\rho_1(-u_1 A)u_1 + \rho_2(u_2 A)u_2 = -(-p_1 A + p_2 A) \quad \Rightarrow p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{4.1.15}$$

Steady Flow Energy Conservation:

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2} \quad \Rightarrow h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4.1.16}$$

Here, the enthalpy $h\left(=e+\dfrac{p}{\rho}\right)$ is defined as another thermodynamic property of the gas.

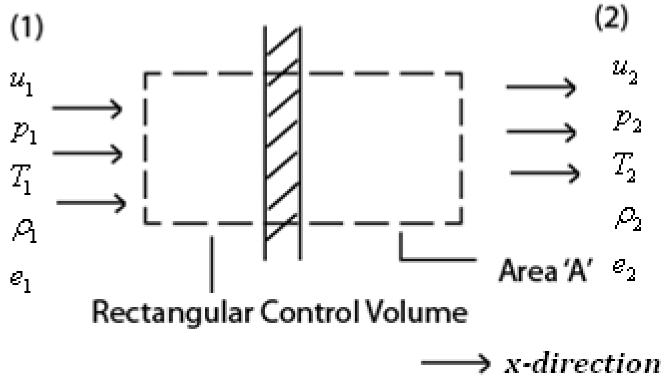

Fig. 4.1.1: Schematic representation of one-dimensional flow.

Module 4 : Lecture 2
COMPRESSIBLE FLOWS
(Fundamental Aspects: Part - II)

Wave Propagation in a Compressible Media

Consider a gas confined in a long tube with piston as shown in Fig. 4.2.1(a). The gas may be assumed to have infinite number of layers and initially, the system is in equilibrium. In other words, the last layer does not feel the presence of piston. Now, the piston is given a very small 'push' to the right. So, the layer of gas adjacent to the piston piles up and is compressed while the reminder of the gas remains unaffected. With due course of time, the compression wave moves downstream and the information is propagated. Eventually, all the gas layers feel the piston movement. If the pressure pulse applied to the gas is small, the wave is called as sound wave and the resultant compression wave moves at the "speed of sound". However, if the fluid is treated as incompressible, the change in density is not allowed. So, there will be no piling of fluid due to instantaneous change and the disturbance is felt at all other locations at the same time. So, the speed of sound depends on the fluid property i.e. 'compressibility'. The lower is its value; more will be the speed of sound. In an ideal incompressible medium of fluid, the speed of sound is infinite. For instance, sound travels about 4.3-times faster in water (1484 m/s) and 15-times as fast in iron (5120 m/s) than air at 20ºC.

Let us analyze the piston dynamics as shown in Fig. 4.2.1(a). If the piston moves at steady velocity dV, the compression wave moves at speed of sound a into the stationary gas. This infinitesimal disturbance creates increase in pressure and density next to the piston and in front of the wave. The same effect can be observed by keeping the wave stationary through dynamic transformation as shown in Fig. 4.2.1 (b). Now all basic one dimensional compressible flow equations can be applied for a very small control enclosing the stationary wave.

<u>Continuity equation</u>: Mass flow rate $(\dot{m})$ is conserved across the stationary wave.

$$\dot{m} = \rho\, a\, A = (\rho + d\rho)(a - dV)A \;\; \Rightarrow dV = \left(\frac{a}{\rho}\right) d\rho \tag{4.2.1}$$

<u>Momentum equation</u>: As long as the compression wave is thin, the shear forces on the control volume are negligibly small compared to the pressure force. The momentum balance across the control volume leads to the following equation;

$$(p + dp)A - pA = \dot{m}\, a - \dot{m}(a - dV) \quad \Rightarrow dV = \left(\frac{1}{\rho a}\right) dp \tag{4.2.2}$$

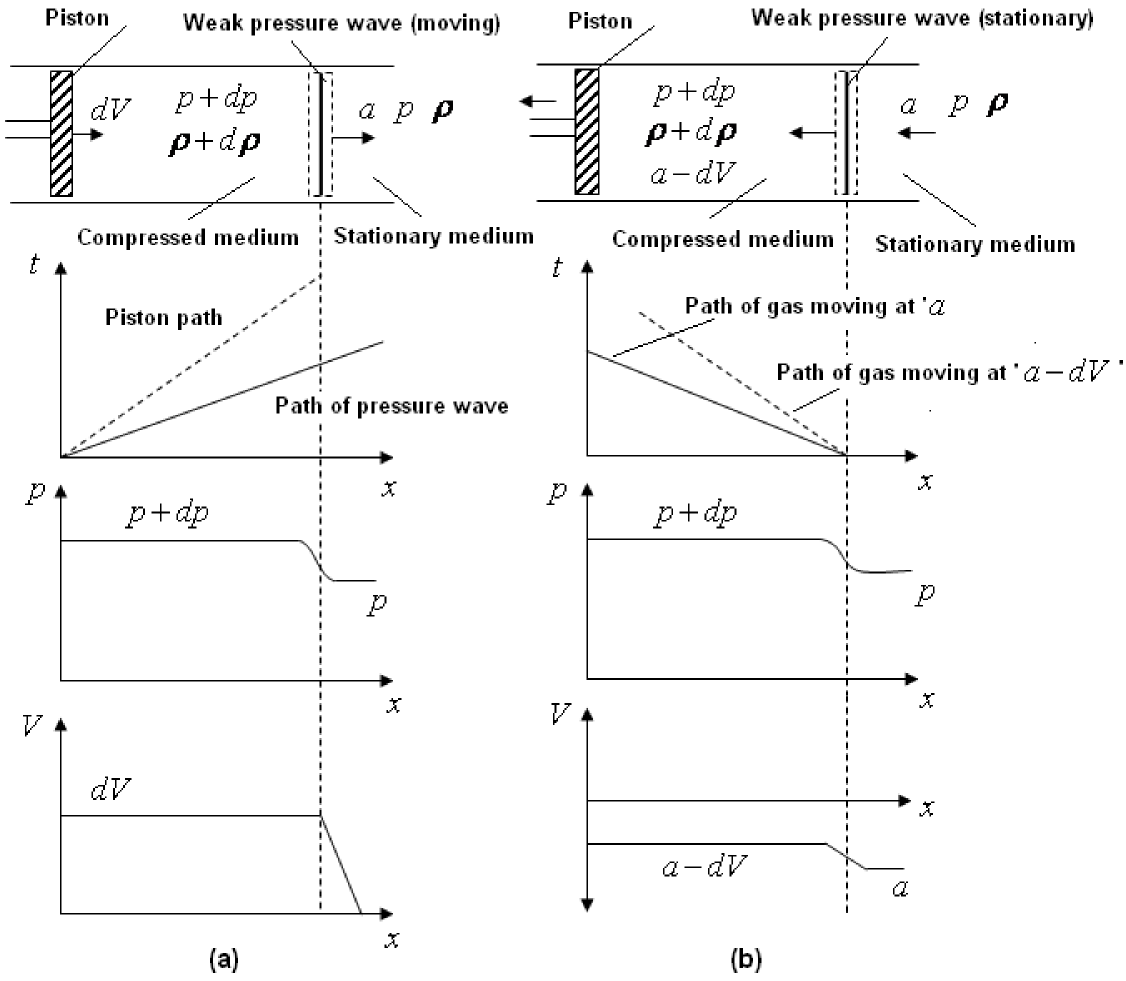

Fig. 4.2.1: Propagation of pressure wave in a compressible medium: (a) Moving wave; (b) Stationary wave.

Energy equation: Since the compression wave is thin, and the motion is very rapid, the heat transfer between the control volume and the surroundings may be neglected and the thermodynamic process can be treated as *adiabatic*. Steady flow energy equation can be used for energy balance across the wave.

$$h+\frac{a^2}{2}=\left(h+dh\right)+\frac{\left(a-dV\right)^2}{2} \quad \Rightarrow dV=\left(\frac{1}{a}\right)dh \tag{4.2.3}$$

Entropy equation: In order to decide the direction of thermodynamic process, one can apply $T-ds$ relation along with Eqs (4.2.2 & 4.2.3) across the compression wave.

$$T\,ds=dh-\frac{dp}{\rho}=0 \quad \Rightarrow ds=0 \tag{4.2.4}$$

Thus, the flow is isentropic across the compression wave and this compression wave can now be called as sound wave. The speed of the sound wave can be computed by equating Eqs.(4.2.1 & 4.2.2).

$$\left(\frac{a}{\rho}\right)=\left(\frac{1}{\rho a}\right) \quad \Rightarrow a^2=\frac{d\,p}{d\rho}=\left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.2.5}$$

Further simplification of Eq. (4.2.5) is possible by evaluating the differential with the use of isenropic equation.

$$\frac{p}{\rho^{\gamma}}=\text{constant} \quad \Rightarrow \ln p-\gamma \ln \rho=\text{constant} \tag{4.2.6}$$

Differentiate Eq. (4.2.6) and apply perfect gas equation $\left(p=\rho RT\right)$ to obtain the expression for speed of sound. is obtained as below;

$$\left(\frac{\partial p}{\partial \rho}\right)_s=\frac{\gamma p}{\rho} \quad \Rightarrow a=\sqrt{\frac{\gamma p}{\rho}}=\sqrt{\gamma RT} \tag{4.2.7}$$

Mach number

It may be seen that the speed of sound is the thermodynamic property that varies from point to point. When there is a large relative speed between a body and the compressible fluid surrounds it, then the compressibility of the fluid greatly influences the flow properties. Ratio of the local speed (V) of the gas to the speed of sound (a) is called as local Mach number (M).

$$M = \frac{V}{a} = \frac{V}{\sqrt{\gamma RT}} \tag{4.2.8}$$

There are few physical meanings for Mach number;

(a) It shows the compressibility effect for a fluid i.e. $M < 0.3$ implies that fluid is incompressible.

(b) It can be shown that Mach number is proportional to the ratio of kinetic to internal energy.

$$\frac{\left(V^2/2\right)}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{RT/(\gamma-1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma-1)} = \frac{\gamma(\gamma-1)}{2} M^2 \tag{4.2.9}$$

(c) It is a measure of directed motion of a gas compared to the random thermal motion of the molecules.

$$M^2 = \frac{V^2}{a^2} = \frac{\text{directed kinetic energy}}{\text{random kinetic energy}} \tag{4.2.10}$$

Compressible Flow Regimes

In order to illustrate the flow regimes in a compressible medium, let us consider the flow over an aerodynamic body (Fig. 4.2.2). The flow is uniform far away from the body with free stream velocity (V_∞) while the speed of sound in the uniform stream is a_∞. Then, the free stream Mach number becomes $M_\infty \left(= V_\infty / a_\infty\right)$. The streamlines can be drawn as the flow passes over the body and the local Mach number can also vary along the streamlines. Let us consider the following distinct flow regimes commonly dealt with in compressible medium.

<u>Subsonic flow</u>: It is a case in which an airfoil is placed in a free stream flow and the local Mach number is less than unity everywhere in the flow field (Fig. 4.2.2-a). The flow is characterized by smooth streamlines with continuous varying properties. Initially, the streamlines are straight in the free stream, but begin to deflect as they approach the body. The flow expands as it passed over the airfoil and the local Mach number on the top surface of the body is more than the free stream value. Moreover, the local Mach number(M) in the surface of the airfoil remains always less than 1, when the free stream Mach number $(M_\infty)$is sufficiently less than 1. This regime is defined as subsonic flow which falls in the range of free stream Mach number less than 0.8 i.e. $M_\infty \leq 0.8$.

<u>Transonic flow</u>: If the free stream Mach number increases but remains in the subsonic range close to 1, then the flow expansion over the air foil leads to supersonic region locally on its surface. Thus, the entire regions on the surface are considered as mixed flow in which the local Mach number is either less or more than 1 and thus called as *sonic pockets* (Fig. 4.2.2-b). The phenomena of sonic pocket is initiated as soon as the local Mach number reaches 1 and subsequently terminates in the downstream with a shock wave across which there is discontinuous and sudden change in flow properties. If the free stream Mach number is slightly above unity (Fig. 4.2.2-c), the shock pattern will move towards the trailing edge and a second shock wave appears in the leading edge which is called as *bow shock*. In front of this bow shock, the streamlines are straight and parallel with a uniform supersonic free stream Mach number. After passing through the bow shock, the flow becomes subsonic close to the free stream value. Eventually, it further expands over the airfoil

surface to supersonic values and finally terminates with trailing edge shock in the downstream. The mixed flow patterns sketched in Figs. 4.2.2 (b & c), is defined as the *transonic regime*.

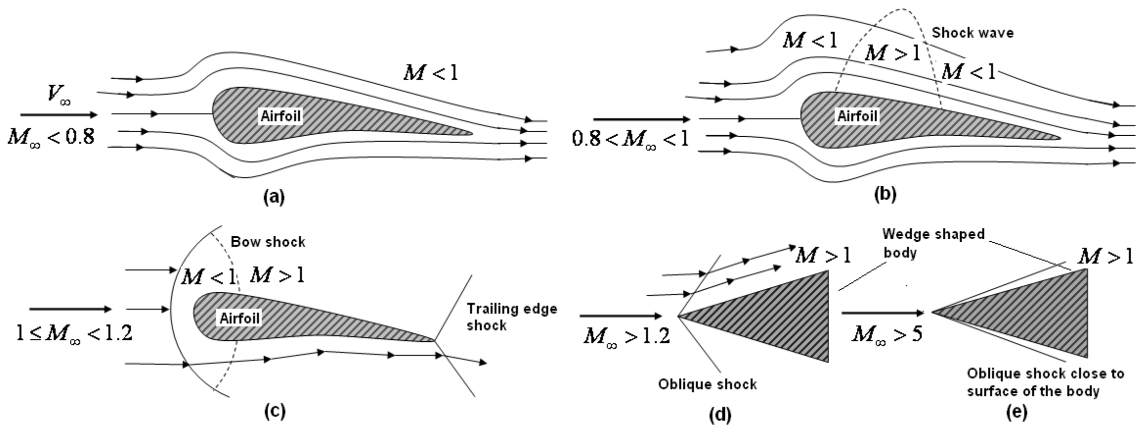

Fig. 4.2.2: Illustration of compressible flow regime: (a) subsonic flow; (b & c) transonic flow; (d) supersonic flow; (d) hypersonic flow.

<u>Supersonic flow</u>: In a flow field, if the Mach number is more than 1 everywhere in the domain, then it defined as supersonic flow. In order to minimize the drag, all aerodynamic bodies in a supersonic flow, are generally considered to be sharp edged tip. Here, the flow field is characterized by straight, oblique shock as shown in Fig. 4.2.2(d). The stream lines ahead of the shock the streamlines are straight, parallel and horizontal. Behind the oblique shock, the streamlines remain straight and parallel but take the direction of wedge surface. The flow is supersonic both upstream and downstream of the oblique shock. However, in some exceptional strong oblique shocks, the flow in the downstream may be subsonic.

<u>Hypersonic flow</u>: When the free stream Mach number is increased to higher supersonic speeds, the oblique shock moves closer to the body surface (Fig. 4.2.2-e). At the same time, the pressure, temperature and density across the shock increase explosively. So, the flow field between the shock and body becomes hot enough to ionize the gas. These effects of thin shock layer, hot and chemically reacting gases and many other complicated flow features are the characteristics of *hypersonic flow*. In reality, these special characteristics associated with hypersonic flows appear gradually as the free stream Mach numbers is increased beyond 5.

As a rule of thumb, the compressible flow regimes are classified as below;

$$M < 0.3 \left(\text{incompressible flow}\right)$$
$$M < 1 \left(\text{subsonic flow}\right)$$
$$0.8 < M < 1.2 \left(\text{transonic flow}\right)$$
$$M > 1 \left(\text{supersonic flow}\right)$$
$$M > 5 \text{ and above } \left(\text{hypersonic flow}\right)$$

Rarefied and Free Molecular Flow: In general, a gas is composed of large number of discrete atoms and molecules and all move in a random fashion with frequent collisions. However, all the fundamental equations are based on overall macroscopic behavior where the continuum assumption is valid. If the mean distance between atoms/molecules between the collisions is large enough to be comparable in same order of magnitude as that of characteristics dimension of the flow, then it is said to be low density/rarefied flow. Under extreme situations, the mean free path is much larger than the characteristic dimension of the flow. Such flows are defined as free molecular flows. These are the special cases occurring in flight at very high altitudes (beyond 100 km) and some laboratory devices such as electron beams.

Module 4 : Lecture 3
COMPRESSIBLE FLOWS
(Isentropic and Characteristics States)

An isentropic process provides the useful standard for comparing various types of flow with that of an idealized one. Essentially, it is the process where all types of frictional effects are neglected and no heat addition takes place. Thus, the process is considered as reversible and adiabatic. With this useful assumption, many fundamental relations are obtained and some of them are discussed here.

Stagnation/Total Conditions

When a moving fluid is decelerated isentropically to reach zero speed, then the thermodynamic state is referred to as *stagnation/total* condition/state. For example, a gas contained in a high pressure cylinder has no velocity and the thermodynamic state is known as *stagnation/total* condition (Fig. 4.3.1-a). In a real flow field, if the actual conditions of pressure (p), temperature (T), density (ρ), enthalpy(h), internal energy (e), entropy (s)etc. are referred to as static conditions while the associated stagnation parameters are denoted as p_0, T_0, ρ_0, h_0, e_0 and s_0, respectively. The stagnation state is fixed by using second law of thermodynamics where $s = s_0$ as represented in *enthalpy-entropy diagram called as the Mollier diagram* (Fig. 4.3.1-b).

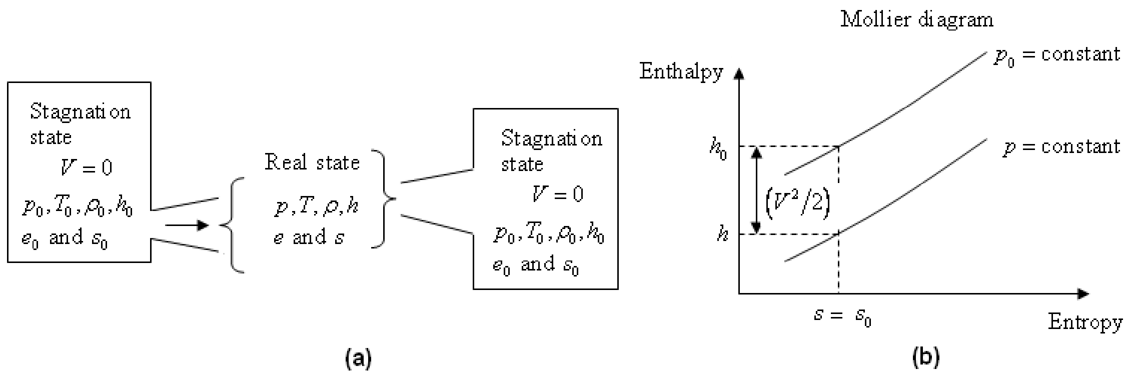

Fig 4.3.1: (a) Schematic representation of stagnation condition; (b) Mollier diagram.

The simplified form of energy equation for steady, one-dimensional flow with no heat addition, across two regions 1 and 2 of a control volume is given by,

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4.3.1}$$

For a calorically perfect gas, replacing, $h = c_p T$, so the Eq. (4.3.1) becomes,

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \tag{4.3.2}$$

If the region '1' refers to any arbitrary real state in the flow field and the region '2' refers to stagnation condition, then Eq. (4.3.2) becomes,

$$c_p T + \frac{u^2}{2} = c_p T_0 \tag{4.3.3}$$

It can be solved for $\left(T_0/T\right)$ as,

$$\begin{aligned} \frac{T_0}{T} &= 1 + \frac{u^2}{2c_p T} = 1 + \frac{u^2}{2\gamma RT/(\gamma-1)} = 1 + \frac{u^2}{2a^2/(\gamma-1)} \\ \text{or, } \frac{T_0}{T} &= 1 + \left(\frac{\gamma-1}{2}\right)\left(\frac{u}{a}\right)^2 = 1 + \left(\frac{\gamma-1}{2}\right)M^2 \end{aligned} \tag{4.3.4}$$

For an isentropic process, the thermodynamic relation is given by,

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} \tag{4.3.5}$$

From, Eqs (4.3.4) and (4.3.5), the following relations may be obtained for stagnation pressure and density.

$$\begin{aligned} \frac{p_0}{p} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} \\ \frac{\rho_0}{\rho} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} \end{aligned} \tag{4.3.6}$$

In general, if the flow field is isentropic throughout, the stagnation properties are constant at every point in the flow. However, if the flow in the regions '1' and '2' is non-adiabatic and irreversibile, then $T_{01} \neq T_{02};\ p_{01} \neq p_{02};\ \rho_{01} \neq \rho_{02}$

Characteristics Conditions

Consider an arbitrary flow field, in which a fluid element is travelling at some Mach number (M) and velocity (V) at a given point 'A'. The static pressure, temperature and density are p,T and ρ, respectively. Now, imagine that the fluid element is adiabatically slowed down $(\text{if } M>1)$ or speeded up $(\text{if } M<1)$ until the Mach number at 'A' reaches the sonic state as shown in Fig. 4.3.2. Thus, the temperature will change in this process. This imaginary situation of the flow field when a real state in the flow is brought to sonic state is known as the *characteristics conditions*. The associated parameters are denoted as p^*, T^*, ρ^*, a^* etc.

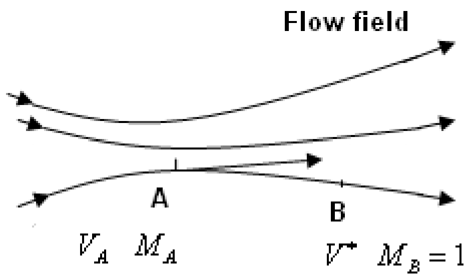

Fig. 4.3.2: Illustration of characteristics states of a gas.

Now, revisit Eq. (4.3.2) and use the relations for a calorically perfect gas, by replacing, $c_p = \dfrac{\gamma R}{\gamma - 1}$ and $a = \sqrt{\gamma RT}$. Another form of energy equation is obtained as below;

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{u_2^2}{2} \tag{4.3.7}$$

At the imagined condition (point 2) of Mach 1, the flow velocity is sonic and $u_2 = a^*$. Then the Eq. (4.3.7) becomes,

$$\begin{aligned} &\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2} \\ &\text{or, } \frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{\gamma+1}{2(\gamma-1)} a^{*2} \end{aligned} \tag{4.3.8}$$

Like stagnation properties, these imagined conditions are associated properties of any fluid element which is actually moving with velocity u. If an actual flow field is non-adiabatic from $A \rightarrow B$, then $a_A^* \neq a_B^*$.On the other hand if the general flow field is adiabatic throughout, then a^* is a constant value at every point in the flow. Dividing u^2 both sides for Eq. (4.3.8) leads to,

$$\frac{(a/u)^2}{\gamma-1}+\frac{1}{2}=\frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{u}\right)^2$$
$$\text{or, } M^2=\frac{2}{\left[(\gamma+1)/M^{*2}\right]-(\gamma-1)} \tag{4.3.9}$$

This equation provides the relation between actual Mach number (M) and *characteristics Mach number* (M^*). It may be shown that when M approaches infinity, M^* reaches a finite value. From Eq. (4.3.9), it may be seen that

$$\begin{aligned} M=1 &\Rightarrow M^*=1 \\ M<1 &\Rightarrow M^*<1 \\ M>1 &\Rightarrow M^*>1 \\ M\rightarrow\infty &\Rightarrow M^*\rightarrow\sqrt{\frac{\gamma+1}{\gamma-1}} \end{aligned} \tag{4.3.10}$$

Relations between stagnation and characteristics state

The stagnation speed and characteristics speed of sound may be written as,

$$a_0=\sqrt{\gamma RT_0};\ a^*=\sqrt{\gamma RT^*} \tag{4.3.11}$$

Rewrite Eq. (4.3.7) for stagnation conditions as given below;

$$\frac{a^2}{\gamma-1}+\frac{u^2}{2}=\frac{a_o^2}{\gamma-1} \tag{4.3.12}$$

Equate Eqs. (4.3.8) and (4.3.12),

$$\frac{\gamma+1}{2(\gamma-1)}a^{*2}=\frac{a_0^2}{\gamma-1}\quad\Rightarrow\left(\frac{a^*}{a_0}\right)^2=\frac{T^*}{T_0}=\frac{2}{\gamma+1} \tag{4.3.13}$$

More useful results may be obtained for Eqs. (4.3.4) & (4.3.6), when we define $p = p^*;\ T = T^*;\ \rho = \rho^*; a = a^*$ for Mach 1

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}};\ \frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \tag{4.3.14}$$

With $\gamma = 1.4$ (for air), the Eqs (4.3.13) & (4.3.14) reduces to constant value.

$$\left(\frac{a^*}{a_0}\right)^2 = \frac{T^*}{T_0} = 0.833;\quad \frac{p^*}{p_0} = 0.528;\ \frac{\rho^*}{\rho_0} = 0.634 \tag{4.3.15}$$

Critical speed and Maximum speed

The critical speed of the gas $\left(u^*\right)$ is same as that speed of sound $\left(a^*\right)$ at sonic state i.e. $u^* = a^*$ at $M = 1$. A gas can attain the maximum speed $\left(u_{\max}\right)$ when it is hypothetically expanded to zero pressure. The static temperature corresponding to this state is also zero. The maximum speed of the gas represents the speed corresponding to the complete transformation of kinetic energy associated with the random motion of gas molecules into the directed kinetic energy. Rearranging Eq. (4.3.3), one can obtain the following equation;

$$\begin{aligned} &T_0 = T + \left(\frac{\gamma-1}{2\gamma R}\right)u^2;\ \text{ At } T = 0;\ \ u = u_{\max} = \sqrt{\frac{2\gamma R T_0}{\gamma-1}} \\ &\text{or, } \left(\frac{u_{\max}}{a_0}\right)^2 = \frac{2}{\gamma-1} \end{aligned} \tag{4.3.16}$$

Now, the Eqs (4.3.13) & (4.3.16) can be simplified to obtain the following relation;

$$\frac{u_{\max}}{a^*} = \sqrt{\frac{\gamma+1}{\gamma-1}} \tag{4.3.17}$$

Steady Flow Adiabatic Ellipse

It is an ellipse in which all the points have same total energies. Each point differs from the other owing to relative proportions of thermal and kinetic energies corresponding to different Mach numbers. Now, rewrite Eq. (4.3.3) by replacing $c_p = \frac{\gamma R}{\gamma - 1}$ and $a = \sqrt{\gamma RT}$;

$$\frac{u^2}{2} + \frac{\gamma R}{\gamma - 1} T = c_1 \quad \Rightarrow u^2 + \left(\frac{2}{\gamma - 1} \right) a^2 = c \tag{4.3.18}$$

When, $T = 0,\ u = u_{\max}$ so that the constant appearing in Eq. (4.3.18) can be considered as, $c = u_{\max}^2$. Then, Eq. (4.3.18) is written as follows;

$$u^2 + \left(\frac{2}{\gamma - 1} \right) a^2 = u_{\max}^2 \quad \Rightarrow \frac{u^2}{u_{\max}^2} + \left(\frac{2}{\gamma - 1} \right) \frac{a^2}{u_{\max}^2} = 1 \tag{4.3.19}$$

Replacing the value of $u_{\max}^2$ from Eq. (4.3.16) in Eq. (4.3.19), one can write the following expression;

$$\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1 \tag{4.3.20}$$

This is the equation of an ellipse with major axis as $u_{\max}$ and minor axis as a_0 as shown in Fig. 4.3.3. Now, rearrange Eq. (4.3.20) in the following form;

$$a^2 = a_0^2 - \left(\frac{u^2}{u_{\max}^2} \right) a_0^2 \tag{4.3.21}$$

Now, differentiate Eq. (4.3.21) with respect to u and simplify;

$$\frac{da}{du} = -\left(\frac{\gamma - 1}{2} \right) \left(\frac{u}{a} \right) = -\left(\frac{\gamma - 1}{2} \right) M \quad \Rightarrow M = -\left(\frac{2}{\gamma - 1} \right) \frac{da}{du} \tag{4.3.22}$$

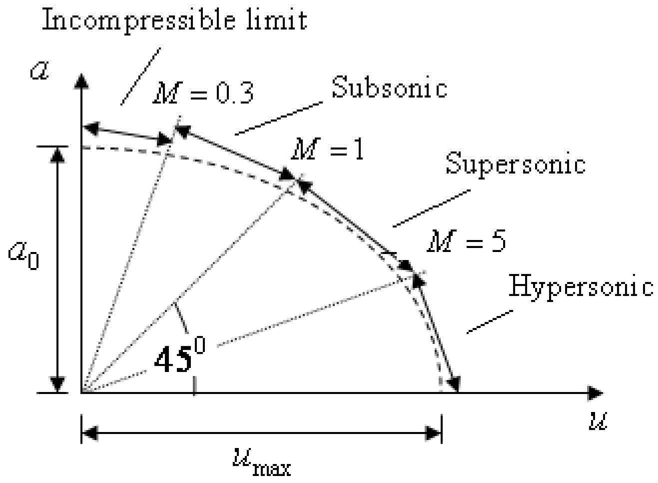

Fig. 4.3.3: Steady flow adiabatic ellipse.

Thus, the change of slope from point to point on the ellipse indicates the change in Mach number and hence the speed of sound and velocity. So, it gives the direct comparison of the relative magnitudes of thermal and kinetic energies. Different compressible flow regimes can be obtained with the knowledge of slope in Fig. 4.3.2. The following important inferences may be drawn;

- In high Mach numbers flows, the changes in Mach number are mainly due to the changes in speed of sound.
- At low Mach numbers flows, the changes in Mach number are mainly due to the changes in the velocity.
- When the flow Mach number is below 0.3, the changes in speed of sound is negligible small and the flow is treated as incompressible.

Module 4 : Lecture 4
COMPRESSIBLE FLOWS
(One-Dimensional Analysis)

Mach Waves

Consider an aerodynamic body moving with certain velocity (V) in a still air. When the pressure at the surface of the body is greater than that of the surrounding air, it results an infinitesimal compression wave that moves at speed of sound (a). These disturbances in the medium spread out from the body and become progressively weaker away from the body. If the air has to pass smoothly over the surface of the body, the disturbances must 'warn' the still air, about the approach of the body. Now, let us analyze two situations: (a) the body is moving at subsonic speed $(V < a; M < 1)$; (b) the body is moving at supersonic speed $(V > a; M > 1)$.

Case I: During the motion of the body, the sound waves are generated at different time intervals (t) as shown in Fig. 4.4.1. The distance covered by the sound waves can be represented by the circle of radius $(at, 2at, 3at.......\text{so on})$. During same time intervals (t), the body will cover distances represented by, $Vt,\ 2Vt,\ 3Vt.......\text{so on}$. At subsonic speeds $(V < a; M < 1)$, the body will always remains inside the family of circular sound waves. In other words, the information is propagated through the sound wave in all directions. Thus, the surrounding still air becomes aware of the presence of the body due to the disturbances induced in the medium. Hence, the flow adjusts itself very much before it approaches the body.

Case II: Consider the case, when the body is moving at supersonic speed $(V > a; M > 1)$. With a similar manner, the sound waves are represented by circle of radius $(at, 2at, 3at.......\text{so on})$ after different time (t) intervals. By this time, the body would have moved to a different location much faster from its initial position. At any point of time, the location of the body is always outside the family of circles of sound waves. The pressure disturbances created by the body always lags behind the body that created the disturbances. In other words, the information reaches the surrounding air much later because the disturbances cannot overtake the body. Hence, the flow cannot adjust itself when it approaches the body. The nature induces a wave across which the flow properties have to change and this line of disturbance is known as "Mach wave". These mach waves are initiated when the speed of the body approaches the speed of sound $(V = a; M = 1)$. They become progressively stronger with increase in the Mach number.

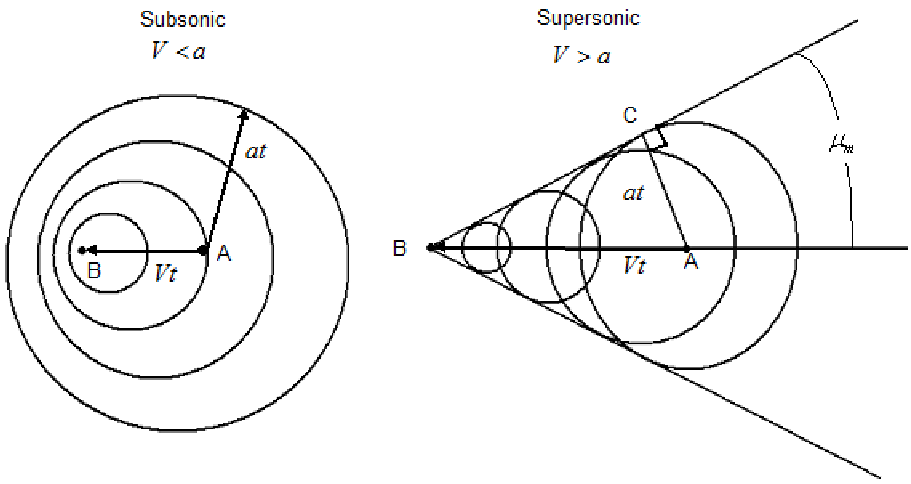

Fig. 4.4.1: Spread of disturbances at subsonic and supersonic speeds.

Some silent features of a *Mach wave* are listed below;

- The series of wave fronts form a disturbance envelope given by a straight line which is tangent to the family of circles. It will be seen that all the disturbance waves lie within a cone (Fig. 4.4.1), having a *vertex/apex* at the body at time considered. The locus of all the leading surfaces of the waves of this cone is known as *Mach cone*.
- All disturbances confine inside the Mach cone extending downstream of the moving body is called as *zone of action*. The region outside the Mach cone and extending upstream is known as *zone of silence*. The pressure disturbances are largely concentrated in the neighborhood of the Mach cone that forms the outer limit of the zone of action (Fig. 4.4.2).

- The half angle of the Mach cone is called as the Mach angle (μ_m) that can be easily calculated from the geometry of the Fig. 4.4.1.

$$\sin\mu_m = \frac{at}{Vt} = \frac{a(2t)}{V(2t)} = \frac{a(3t)}{V(3t)}....... = \frac{a}{V} = \frac{1}{M} \;\Rightarrow \mu_m = \sin^{-1}\left(\frac{1}{M}\right) \qquad (4.4.1)$$

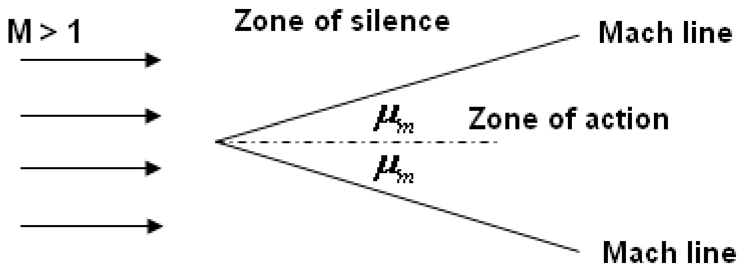

Fig. 4.4.2: Illustration of a Mach wave.

Shock Waves

Let us consider a subsonic and supersonic flow past a body as shown in Fig. 4.3.3. In both the cases, the body acts as an obstruction to the flow and thus there is a change in energy and momentum of the flow. The changes in flow properties are communicated through pressure waves moving at speed of sound everywhere in the flow field (i.e. both upstream and downstream). As shown in Fig. 4.3.3(a), if the incoming stream is subsonic i.e. $M_\infty < 1; \quad V_\infty < a_\infty$, the sound waves propagate faster than the flow speed and warn the medium about the presence of the body. So, the streamlines approaching the body begin to adjust themselves far upstream and the flow properties change the pattern gradually in the vicinity of the body. In contrast, when the flow is supersonic, (Fig. 4.3.3-b) i.e. $M_\infty > 1; \quad V_\infty > a_\infty$, the sound waves overtake the speed of the body and these weak pressure waves merge themselves ahead of the body leading to compression in the vicinity of the body. In other words, the flow medium gets compressed at a very short distance ahead of the body in a very thin region that may be comparable to the mean free path of the molecules in the medium. Since, these compression waves propagate upstream, so they tend to merge as *shock wave*. Ahead of the shock wave, the flow has no idea of presence of the body and immediately behind the shock; the flow is subsonic as shown in Fig. 4.3.3(b).

The thermodynamic definition of a shock wave may be written as "the instantaneous compression of the gas". The energy for compressing the medium, through a shock wave is obtained from the kinetic energy of the flow upstream the shock wave. The reduction in kinetic energy is accounted as heating of the gas to a static temperature above that corresponding to the isentropic compression value. Consequently, in flowing through the shock wave, the gas experiences a decrease in its available energy and accordingly, an increase in entropy. So, the compression through a shock wave is considered as an irreversible process.

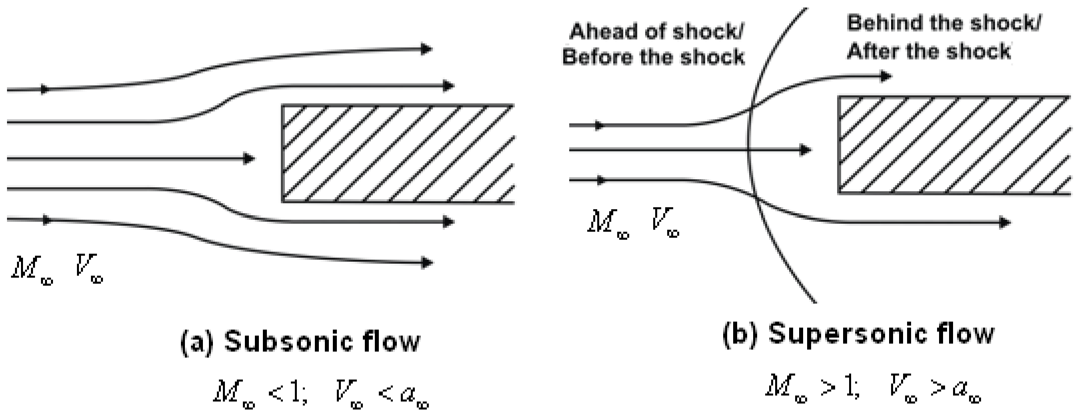

Fig. 4.4.3: Illustration of shock wave phenomena.

Normal Shock Waves

A normal shock wave is one of the situations where the flow properties change drastically in one direction. The shock wave stands perpendicular to the flow as shown in Fig. 4.4.4. The quantitative analysis of the changes across a normal shock wave involves the determination of flow properties. All conditions of are known ahead of the shock and the unknown flow properties are to be determined after the shock. There is no heat added or taken away as the flow traverses across the normal shock. Hence, the flow across the shock wave is adiabatic $\left(\dot{q}=0\right)$.

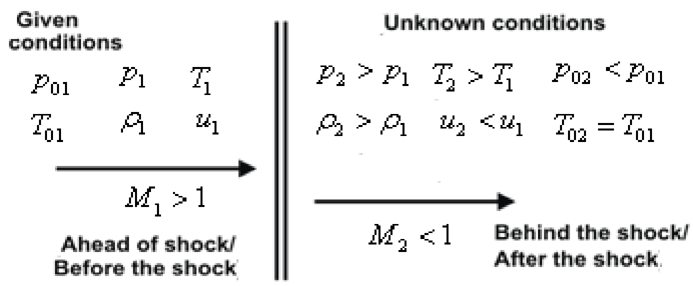

Fig. 4.4.4: Schematic diagram of a standing normal shock wave.

The basic one dimensional compressible flow equations can be written as below;

$$\rho_1 u_1 = \rho_2 u_2; \quad p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2; \quad h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4.4.2}$$

For a calorically perfect gas, thermodynamic relations can be used,

$$p = \rho RT; \; h = c_p T; \; a = \sqrt{\gamma p / \rho} \tag{4.4.3}$$

The continuity and momentum equations of Eq. (4.4.2) can be simplified to obtain,

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \tag{4.4.4}$$

Since, $a^* = \sqrt{\gamma RT^*}$ and $M^* = \frac{V}{a^*}$, the energy equation is written as,

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} \;\Rightarrow a^2 = \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma - 1}{2} u^2 \tag{4.4.5}$$

Both a_1^2 and a_2^2 can now be expressed as,

$$a_1^2 = \frac{\gamma + 1}{2}\left(a^*\right)^2 - \frac{\gamma - 1}{2} u_1^2; \quad a_2^2 = \frac{\gamma + 1}{2}\left(a^*\right)^2 - \frac{\gamma - 1}{2} u_2^2 \tag{4.4.6}$$

Substitute Eqs. (4.4.6) in Eq. (4.4.4) and solve for a^{*2}

$$a^{*2} = u_1 u_2 \;\Rightarrow M_2^* = \frac{1}{M_1^*} \tag{4.4.7}$$

Recall the relation for M and M^* and substitute in Eq. (4.4.7),

$$M^{*2} = \frac{\left(\gamma + 1\right) M^2}{2 + \left(\gamma + 1\right) M^2} \tag{4.4.8}$$

Substitute Eq. (4.4.8) in Eq. (4.4.7) and solve for M_2

$$M_2^2 = \frac{1 + \left(\frac{\gamma - 1}{2}\right) M_1^2}{\gamma M_1^2 - \left(\frac{\gamma - 1}{2}\right)} \tag{4.4.9}$$

Using continuity equation and Prandtl relation, we can write,

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} = \left(M_1^*\right)^2 \tag{4.4.10}$$

Substitute Eq. (4.4.8) in Eq. (4.4.10) and solve for density and velocity ratio across the normal shock.

$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}=\frac{(\gamma+1)M_1^{\,2}}{2+(\gamma-1)M_1^{\,2}} \tag{4.4.11}$$

The pressure ratio can be obtained by the combination of momentum and continuity equations i.e.

$$p_2-p_1=\rho_1 u_1(u_1-u_2)=\rho_1 u_1^2\left(1-\frac{u_2}{u_1}\right);\ \Rightarrow\frac{p_2-p_1}{p_1}=\gamma M_1^2\left(1-\frac{u_2}{u_1}\right) \tag{4.4.12}$$

Substituting the ratio $\left(\frac{u_1}{u_2}\right)$ from Eq. (4.4.10) in Eq. (4.4.12) and simplifying for the pressure ratio across the normal shock, we get,

$$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right) \tag{4.4.13}$$

For a calorically perfect gas, equation of state relation (Eq. 4.4.3) can be used to obtain the temperature ratio across the normal shock i.e.

$$\frac{h_2}{h_1}=\frac{T_2}{T_1}=\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)=\left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right]\left[\frac{2+(\gamma-1)M_1^{\,2}}{(\gamma+1)M_1^{\,2}}\right] \tag{4.4.14}$$

Thus, the upstream Mach number is the powerful tool to dictating the shock wave properties. The “stagnation properties” across the normal shock can be computed as follows;

$$\frac{p_{02}}{p_{01}}=\frac{(p_{02}/p_2)}{(p_{01}/p_1)}\left(\frac{p_2}{p_1}\right) \tag{4.4.15}$$

Here, the ratios $\left(\frac{p_{01}}{p_1}\right)$ and $\left(\frac{p_{02}}{p_2}\right)$ can be obtained from the isentropic relation for the regions '1 and 2' respectively. Knowing the upstream Mach number M_1, Eq. (4.4.9) gives the downstream Mach number M_2. Further, Eq. (4.4.13) can be used to obtain the static pressure ratio $\left(\frac{p_2}{p_1}\right)$. After substitution of these ratios, Eq. (4.4.15) reduces to,

$$\frac{p_{02}}{p_{01}}=\frac{\left(1+\frac{\gamma-1}{2}M_2^{\,2}\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_1^{\,2}\right)^{\frac{\gamma}{\gamma-1}}}\left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right] \tag{4.4.16}$$

Many a times, another significant pressure ratio $\left(\frac{p_{02}}{p_1}\right)$ is important for a normal shock which is normally called as *Rayleigh Pitot Tube* relation.

$$\frac{p_{02}}{p_1}=\left(\frac{p_{02}}{p_2}\right)\left(\frac{p_2}{p_1}\right)\Rightarrow\frac{p_{02}}{p_1}=\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}\left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right] \tag{4.4.17}$$

Recall the energy equation for a calorically perfect gas:

$$c_pT_1+\frac{u_1^2}{2}=c_pT_2+\frac{u_2^2}{2}\;\;\Rightarrow c_pT_{01}=c_pT_{01} \tag{4.4.18}$$

Thus, the stagnation temperatures do not change across a normal shock.

Entropy across a normal shock

The compression through a shock wave is considered as irreversible process leading to an increase in entropy. The change in entropy can be written as a function of static pressure and static temperature ratios across the normal shock.

$$s_2-s_1=c_p\ln\left(\frac{T_2}{T_1}\right)-R\ln\left(\frac{p_2}{p_1}\right) \tag{4.4.19}$$

Mathematically, it can be seen that the entropy change across a normal shock is also a function of the upstream Mach number. The *second law of thermodynamics* puts the limit that 'entropy' must increase $\left(s_2-s_1\geq 0\right)$ for a process to occur in a certain direction. Hence, the upstream Mach number (M_1) must be greater than 1 (i.e. supersonic). It leads to the fact that $M_2 \leq 1;\ (p_2/p_1) \geq 1;\ (\rho_2/\rho_1) \geq 1;\ (T_2/T_1) \geq 1$.

The entropy change across a normal shock can also be calculated from another simple way by expressing the thermodynamic relation in terms of total pressure. Referring to Fig. 4.4.4, it is seen that the discontinuity occurs only in the thin region across the normal shock. If the fluid elements is brought to rest isentropically from its real state (for both upstream and downstream conditions), then they will reach an imaginary state '1a and 2a'. The expression for entropy change between the imaginary states can be written as,

$$s_{2a} - s_{1a} = c_p \ln\left(\frac{T_{2a}}{T_{1a}}\right) - R\ln\left(\frac{p_{2a}}{p_{1a}}\right) \tag{4.4.20}$$

Since, $s_{2a} = s_2; s_{1a} = s_1; T_{2a} = T_{1a} = T_0;\ p_{2a} = p_{02}$ and $p_{1a} = p_{01}$, the Eq.(4.4.20) reduces to,

$$s_2 - s_1 = -R\ln\left(\frac{p_{02}}{p_{01}}\right) \quad \Rightarrow \frac{p_{02}}{p_{01}} = e^{-(s_2 - s_1)/R} \tag{4.4.21}$$

Because of the fact $s_2 > s_1$, Eq. (4.4.21) implies that $p_{02} < p_{01}$. Hence, the stagnation pressure always decreases across a normal shock.

Module 4 : Lecture 5
COMPRESSIBLE FLOWS
(Two-Dimensional Analysis)

Oblique Shock Wave

The normal shock waves are straight in which the flow before and after the wave is normal to the shock. It is considered as a special case in the general family of oblique shock waves that occur in supersonic flow. In general, oblique shock waves are straight but inclined at an angle to the upstream flow and produce a change in flow direction as shown in Fig. 4.5.1(a). An infinitely weak oblique shock may be defined as a *Mach wave* (Fig. 4.5.1-b). By definition, an oblique shock generally occurs, when a supersonic flow is 'turned into itself" as shown in Fig. 4.5.1(c). Here, a supersonic flow is allowed to pass over a surface, which is inclined at an angle (θ) to the horizontal. The flow streamlines are deflected upwards and aligned along the surface. Since, the upstream flow is supersonic; the streamlines are adjusted in the downstream an oblique shock wave angle (β) with the horizontal such that they are parallel to the surface in the downstream. All the streamlines experience same deflection angle across the oblique shock.

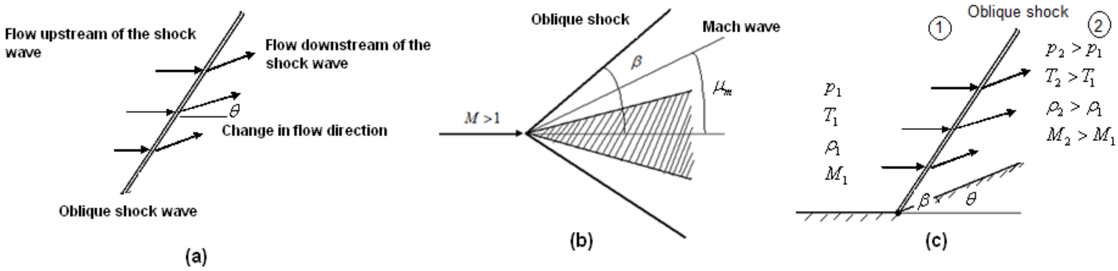

Fig. 4.5.1: Schematic representation of an oblique shock.

Oblique Shock Relations

Unlike the normal shocks, the analysis of oblique shocks is prevalent mainly in the two-dimensional supersonic flows. The flow field properties are the functions of x and y as shown in Fig. 4.5.2. In the upstream of the shock, the streamlines are horizontal where, the Mach number and velocity of the flow are M_1 and V_1, respectively. The flow is deflected towards the shock in the downstream by angle θ such that the Mach number and velocity becomes M_2 and V_2, respectively. The components of V_1, parallel and perpendicular to the shock are u_1 and v_1, respectively. Similarly, the analogous components for V_2 are, u_2 and v_2 respectively. The normal and tangential Mach numbers ahead of the shock are M_{n1} and M_{t1} while the corresponding Mach numbers behind the shock are, M_{n2} and M_{t2} respectively.

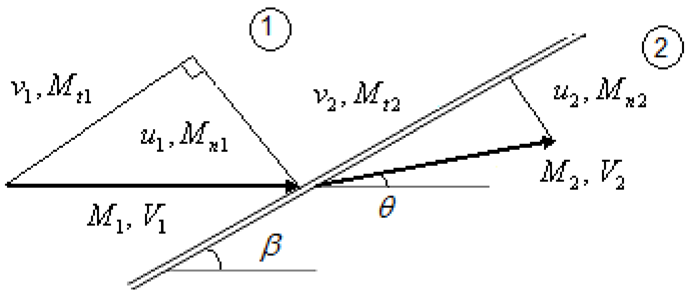

Fig. 4.5.2: Geometrical representation of oblique shock wave.

The continuity equation for oblique shock is,

$$\rho_1 u_1 = \rho_2 u_2 \tag{4.5.1}$$

Considering steady flow with no body forces, the momentum equation can be resolved in tangential and normal directions.

$$\begin{aligned} &\text{Tangential component: } (-\rho_1 u_1) v_1 + (\rho_2 u_2) v_2 = 0 \\ &\text{Normal component: } (-\rho_1 u_1) u_1 + (\rho_2 u_2) u_2 = -(-p_1 + p_2) \end{aligned} \tag{4.5.2}$$

Substitute Eq. (4.5.1) in Eq. (4.5.2),

$$v_1 = v_2; \;\; p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{4.5.3}$$

Thus, it is seen that the tangential component of flow velocity does not change across an oblique shock.

Finally, the energy equation gives,

$$-\left(-p_1u_1+p_2u_2\right)=-\rho_1\left(e_1+\frac{V_1^2}{2}\right)u_1+\rho_2\left(e_1+\frac{V_2^2}{2}\right)u_2 \;\Rightarrow\left(h_1+\frac{V_1^2}{2}\right)=\left(h_2+\frac{V_2^2}{2}\right) \tag{4.5.4}$$

From the geometry of the Fig. 4.5.2, $V^2=u^2+v^2$ and $v_1=v_2$, hence

$$V_1^2-V_2^2=\left(u_1^2+v_1^2\right)-\left(u_2^2+v_2^2\right)=u_1^2-u_2^2 \tag{4.5.5}$$

So, the energy equation becomes,

$$\left(h_1+\frac{u_1^2}{2}\right)=\left(h_2+\frac{u_2^2}{2}\right) \tag{4.5.6}$$

Examining the Eqs (4.5.1, 4.5.3 and 4.5.6), it is noted that they are identical to governing equations for a normal shock. So, the flow properties changes in the oblique shock are governed by the normal component of the upstream Mach number. So, the similar expressions can be written across an oblique shock in terms of normal component of free stream velocity i.e.

$$\begin{aligned}
&M_{n1}=M_1\sin\beta;\quad M_{n2}^2=\left(\frac{M_{n2}^2+\left[2/\left(\gamma-1\right)\right]}{\left[2\gamma/\left(\gamma-1\right)\right]M_{n1}^2-1}\right)\\
&\frac{\rho_2}{\rho_1}=\frac{\left(\gamma+1\right)M_{n1}{}^2}{2+\left(\gamma-1\right)M_{n1}{}^2};\ \frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}\left(M_{n1}^2-1\right);\ \frac{T_2}{T_1}=\frac{p_2}{p_1}\frac{\rho_1}{\rho_2}\\
&M_2=\frac{M_{n2}}{\sin\left(\beta-\theta\right)};\quad s_2-s_1=c_p\ln\left(\frac{T_2}{T_1}\right)-R\ln\left(\frac{p_2}{p_1}\right)
\end{aligned} \tag{4.5.7}$$

Thus, the changes across an oblique shock are function of upstream Mach number $\left(M_1\right)$ and oblique shock angle $\left(\beta\right)$ while the normal shock is a special case when $\beta=\dfrac{\pi}{2}$.

Referring to geometry of the oblique shock (Fig. 4.5.2-b),

$$\tan\beta = \frac{u_1}{v_1};\ \tan(\beta-\theta) = \frac{u_2}{v_2} \tag{4.5.8}$$

Since, $v_1 = v_2$, Eq. (4.5.8) reduces to,

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} \tag{4.5.9}$$

Use the relations given in Eq. (4.5.7) and substituting them in Eq. (4.5.9), the trigonometric equation becomes,

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right] \tag{4.5.10}$$

It is a famous relation showing θ as the unique function of β and M_1. Eq. (4.5.10) is used to obtain the $\theta-\beta-M$ curve (Fig. 4.5.3) for $\gamma = 1.4$.

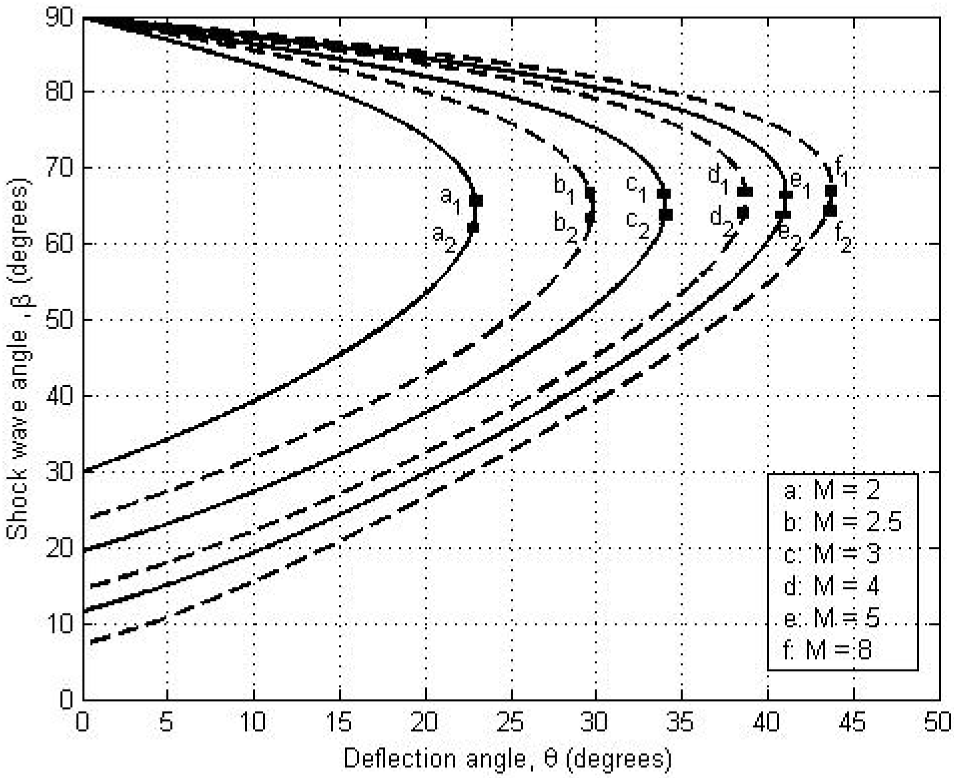

Fig. 4.5.3: $\theta-\beta-M$ **curves for an oblique shock.**

The following inferences may be drawn from $\theta-\beta-M$ curves. It is seen that there is a maximum deflection angle $\theta_{\max}$.

- For any given M_1, if, $\theta<\theta_{\max}$, the oblique shock will be attached to the body (Fig. 4.5.4-a). When $\theta>\theta_{\max}$, there will be no solution and the oblique shock will be curved and detached as shown in Fig. 4.5.4(b). The locus of $\theta_{\max}$ can be obtained by joining the points (a_1, b_1, c_1, d_1, e_1 and f_1) in the Fig. 4.5.3.
- Again, if $\theta<\theta_{\max}$, there will be two values of β predicted from $\theta-\beta-M$ relation. Large value of β corresponds to strong shock solution while small value refers to weak shock solution (Fig. 4.5.4-c). In the strong shock solution, M_2 is subsonic while in the weak shock region, M_2 is supersonic. The locus of such points (a_2, b_2, c_2, d_2, e_2 and f_2) as shown in Fig. 4.5.3, is a curve that also signifies the weak shock solution. The conditions behind the shock could be subsonic if θ becomes closer to $\theta_{\max}$.
- If $\theta=0$, it corresponds to a normal shock when $\beta=\dfrac{\pi}{2}$ and the oblique shock becomes a Mach wave when $\beta=\mu_m$.

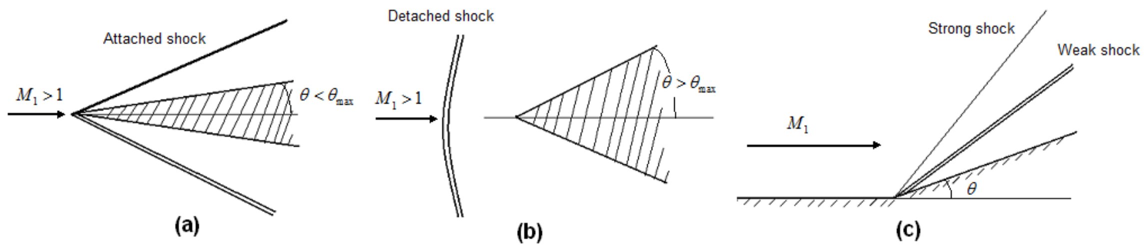

Fig. 4.5.4: (a) Attached shock; (b) Detached shock; (c) Strong and weak shock.

Oblique Expansion Waves

Another class of two dimensional waves occurring in supersonic flow shows the opposite effects of oblique shock. Such types of waves are known as *expansion waves*. When the supersonic flow is "turned away from itself", an expansion wave is formed as shown in Fig. 4.5.5(a). Here, the flow is allowed to pass over a surface which is inclined at an angle (θ) to the horizontal and all the flow streamlines are deflected downwards. The change in flow direction takes place across an expansion fan centered at point 'A'. The flow streamlines are smoothly curved till the downstream flow becomes parallel to the wall surface behind the point 'A'. Here, the flow properties change smoothly through the expansion fan except at point 'A'. An infinitely strong oblique expansion wave may be called as a *Mach wave*. An expansion wave emanating from a sharp convex corner is known as a *centered expansion* which is commonly known as *Prandtl-Meyer expansion wave*. Few features of PM expansion waves are as follows;

- Streamlines through the expansion wave are smooth curved lines.
- The expansion of the flow takes place though an infinite number of Mach waves emitting from the center 'A'. It is bounded by forward and rearward Mach lines as shown in Fig. 4.5.5(b). These Mach lines are defined by Mach angles i.e.

$$\begin{aligned} &\text{Forward Mach angle: } \mu_{m1} = \sin^{-1}\left(1/M_1\right) \\ &\text{Rearward Mach angle: } \mu_{m2} = \sin^{-1}\left(1/M_2\right) \end{aligned} \tag{4.5.11}$$

- The expansion takes place through a continuous succession of Mach waves such that there is no change in entropy for each Mach wave. Thus, the expansion process is treated as isentropic.
- The Mach number increases while the static properties such as pressure, temperature and density decrease during the expansion process.

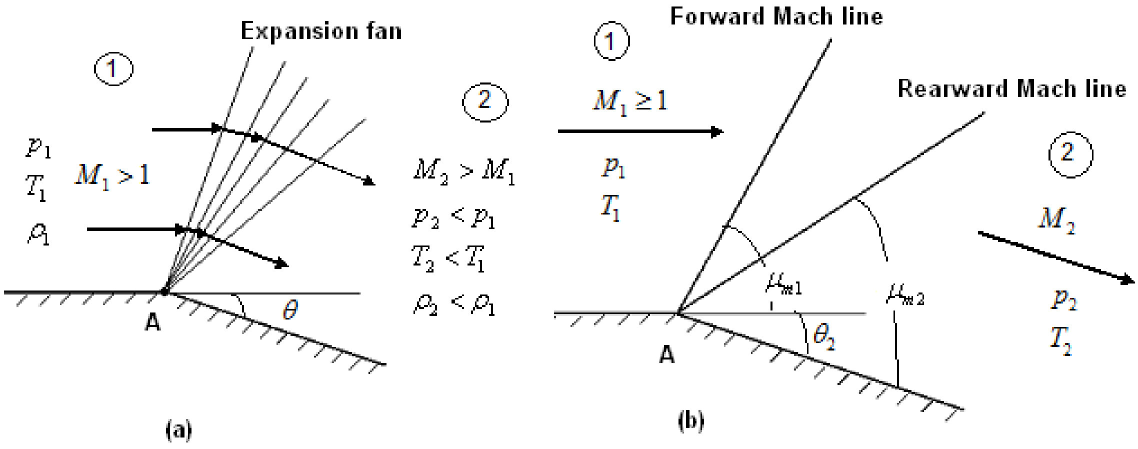

Fig. 4.5.5: Schematic representation of an expansion fan.

The quantitative analysis of expansion fan involves the determination of M_2, p_2, T_2 and ρ_2 for the given upstream conditions of M_1, p_1, T_1, ρ_1 and θ_2. Consider the infinitesimal changes across a very weak wave (Mach wave) as shown in Fig. 4.5.6.

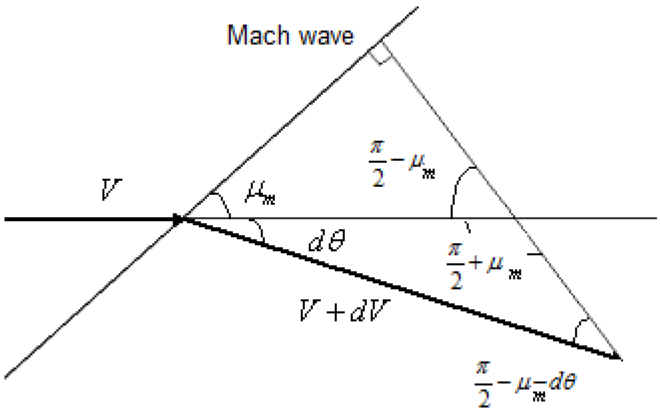

Fig. 4.5.6: Infinitesimal change across a Mach wave.

From the law of sine,

$$\frac{V+dV}{V}=1+\frac{dV}{V}=\frac{\sin\left(\frac{\pi}{2}+\mu_m\right)}{\sin\left(\frac{\pi}{2}-\mu_m-d\theta\right)} \tag{4.5.12}$$

Use trigonometric identities and Taylor series expansion, Eq. (4.5.12) can be simplified as below;

$$d\theta = \frac{\left(dV/V\right)}{\tan\mu_m} \tag{4.5.13}$$

Since, $\sin\mu_m = \frac{1}{M} \Rightarrow \tan\mu_m = \frac{1}{\sqrt{M^2-1}}$, so the Eq. (4.5.13) can be simplified and integrated further from region '1' to '2',

$$d\theta = \sqrt{M^2-1}\frac{dV}{V} \quad \Rightarrow \int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\frac{dV}{V} \tag{4.5.14}$$

From the definition of Mach number,

$$V = Ma \;\; \Rightarrow \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \tag{4.5.15}$$

For a calorically perfect gas, the energy equation can be written as,

$$\left(\frac{a_o}{a}\right)^2 = 1 + \frac{\gamma-1}{2}M^2 \;\; \Rightarrow \frac{da}{a} = -\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1} dM \tag{4.5.16}$$

Use Eqs (4.5.15 & 4.5.16) in Eq. (4.5.14) and integrate from $\theta = 0$ to θ_2,

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - 0 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{4.5.17}$$

The integral in the Eq. (4.5.18) is known as *Prandtl-Meyer function*, $\nu(M)$.

$$\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} = \sqrt{\frac{\gamma+1}{\gamma-1}}\left[\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right] - \tan^{-1}\sqrt{M^2-1} \tag{4.5.18}$$

Finally, Eq. (4.5.17) reduces to,

$$\theta_2 = \nu(M_2) - \nu(M_1) \tag{4.5.19}$$

Thus, for a given upstream Mach number M_1, one can obtain $\nu(M_1)$, subsequently calculate using given $\nu(M_2)$ and θ_2. Since, the expansion process is isentropic, the flow properties can be calculated from isentropic relations.

Module 4 : Lecture 6
COMPRESSIBLE FLOWS
(Hypersonic Flow: Part - I)

Introduction to Hypersonic Flow

The hypersonic flows are different from the conventional regimes of supersonic flows. As a rule of thumb, when the Mach number is greater than 5, the flow is classified as hypersonic. However, the flow does not change its feature all of a sudden during this transition process. So, the more appropriate definition of hypersonic flow would be regime of the flow where certain physical flow phenomena become more important with increase in the Mach number. One of the physical meanings may be given to the Mach number as the measure of the ordered motion of the gas to the random thermal motion of the molecules. In other words, it is the ratio of ordered energy to the random energy as given in Eq. (4.6.1).

$$\mathrm{M}^2=\frac{(1/2)\mathrm{V}^2}{(1/2)\mathrm{a}^2}=\frac{\text{Ordered kinetic energy}}{\text{Random kinetic energy}} \qquad (4.6.1)$$

In the case of hypersonic flows, it is the directed/ordered kinetic energy that dominates over the energy associated with random motion of the molecules. Now, recall the energy equation expressed in the form of flow velocity (V), speed of sound $\left(a=\sqrt{\gamma RT}\right)$ and stagnation speed of sound $\left(a_0=\sqrt{\gamma RT_0}\right)$.

$$\frac{a_0^2}{\gamma-1}=\frac{a^2}{\gamma-1}+\frac{V^2}{2} \quad \Rightarrow \left(\frac{a}{a_0}\right)^2+\left(\frac{\gamma-1}{2}\right)\left(\frac{V}{a_0}\right)^2=1 \qquad (4.6.2)$$

Eq. (4.6.2) forms an adiabatic ellipse which is obtained for steady flow energy equation. When the flow approaches the hypersonic limit, the ratio becomes $\frac{a}{a_0} \ll 1$.

Then, Eq. (4.6.2) simplifies to the following expression.

$$V^2\approx\frac{2a_0^2}{\gamma-1}\approx\frac{2\gamma RT_0}{\gamma-1} \qquad (4.6.3)$$

In other words, the entire kinetic energy of the flow gets converted to internal energy of the flow which is a function total temperature (T_0) of the flow.

The study/research on hypersonic flows revels many exciting and unknown flow features of aerospace vehicles in the twenty-first century. The presence of special features in a hypersonic flow is highly dependent on type of trajectory, configuration of the vehicle design, mission requirement that are decided by the nature of hypersonic atmosphere encountered by the flight vehicle. Therefore, the hypersonic flight vehicles are classified in four different types, based on the design constraints imposed from mission specifications.

- Reentry vehicles (uses the rocket propulsion system)
- Cruise and acceleration vehicle (air-breathing propulsion such as ramjet/scramjet)
- Reentry vehicles (uses both air-breathing and rocket propulsion)
- Aero-assisted orbit transfer vehicle (presence of ions and plasma in the vicinity of spacecraft)

Characteristics Features of Hypersonic Flow

There are certain physical phenomena that essentially differentiate the hypersonic flows as compared to the supersonic flows. Even though, the flow is treated as supersonic, there are certain special features that appear when the speed of the flow is more than the speed of sound typically beyond the Mach number of 5. Some of these characteristics features are listed here;

<u>Thin shock layer</u>: It is known from oblique shock relation $(\theta-\beta-M)$ that the shock wave angle (β) decreases with increase in the Mach number (M) for weak shock solution. With progressive increase in the Mach number, the shock wave angle reaches closer to the flow deflection angle (θ). Again, due to increase in temperature rise across the shock wave, if chemical reaction effects are included, the shock wave angle will still be smaller. Since, the distance between the body and the shock wave is small, the increase in the density across the shock wave results in very high mass fluxes squeezing through small areas. The flow region between the shock wave and the body is known as *thin shock layer* as shown in Fig. 4.6.1(a). It is the basic characteristics of hypersonic flows that shock waves lie closer to the body and shock layer is thin. Further, the shock wave merges with the thick viscous boundary layer growing from the body surface. The complexity of flow field increases due to thin

shock layer where the boundary layer thickness and shock layer thickness become comparable.

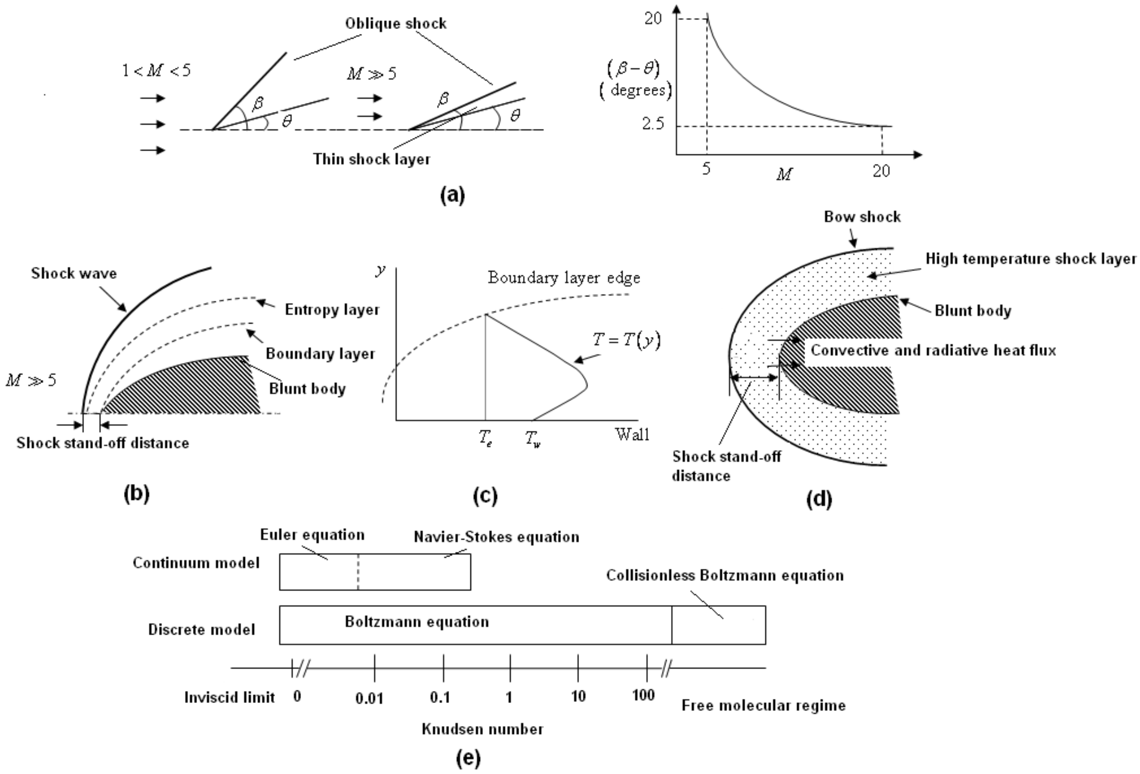

Fig. 4.6.1: Few important phenomena in a hypersonic flow: (a) Thin shock layer; (b) Entropy layer; (c) Temperature profile in a boundary layer; (d) High temperature shock layer; (e) Low density effects.

<u>Entropy layer</u>: The aerodynamic body configuration used in hypersonic flow environment is typically blunt to avoid thin shock layers to be closer to the body. So, there will be a detached bow shock standing at certain distance from the nose of the body and this shock wave is highly curved (Fig. 4.6.1-b). Since, the flow process across the shock is a non-isentropic phenomena, an entropy gradient is developed that varies along the distance of the body. At the nose portion of the blunt body, the bow shock resembles normal to the streamline and the centerline of the flow will experience a larger entropy gradient while all other neighboring streamlines undergo the entropy changes in the weaker portion of the shock. It results in an *entropy layer* that persists all along the body. Using the classical *Crocco's theorem*, the entropy layer may be related to vorticity. Hence, the entropy layer in high Mach number flows, exhibits strong gradient of entropy which leads to higher vorticity at higher magnitudes. Due to the presence of entropy layer, it becomes difficult to predict the boundary layer properties. This phenomenon in the hypersonic flow is called as *vortcity generation.* In addition to thin shock layer, the entropy layer also interacts

with viscous boundary layer that leads to very complicated and unknown flow features.

Viscous-Inviscid interaction: When a high velocity, hypersonic flow is slowed down in the vicinity of the aerodynamic body due to viscous effects within the boundary layer, the major portion of the kinetic energy is transformed into the internal energy of the gas known as *viscous dissipation* leading to increase in temperature. For a cold wall, the typical temperature profile in a boundary layer is shown in Fig. 4.6.1(c). Since, the pressure is constant in the normal direction through the boundary layer, the increase in temperature results decrease in density. In order to pass through a given mass flux at reduced density, the thickness of the boundary layer must be larger. Thus, the displacement thickness increases, causing the body shape to appear much thicker and displacing outer inviscid flow. Hence, the free stream flow encounters an inflated object which changes the shock shape and in turn boundary layer parameters such as surface pressure, wall heat flux, skin friction etc. Again, when the boundary layer becomes thick, it essentially merges with the thin shock layer. Thus, there are major interactions of viscous boundary layer, thin shock layer and outer inviscid flows. This phenomenon is known as viscous-inviscid interaction and has important effect on the surface pressures and the stability of hypersonic vehicles.

High temperature effects: The kinetic energy of the high speed, hypersonic flow is dissipated by the effect of friction within the boundary layer (Fig. 4.6.1-d). The extreme viscous dissipation can result in substantial increase in temperature (~10000 K) exciting the vibration within the molecules and can cause dissociation, ionization in the gas. Typically, in the range of 2000K-4000K, the oxygen molecules start dissociating and with increase in temperature, dissociation of nitrogen molecules takes place. Further increase in temperature (> 9000 K), ionization of both oxygen and nitrogen can start. This leads to chemical reaction within the boundary layer. As a result, the gases within the boundary layer will have variable specific heat ratio and gas constant which are functions of both temperature and pressure. Therefore treatment of air or any fluid flowing with hypersonic speed over any configuration should be done properly by incorporating all the microscopic changes which essentially leads to change in thermodynamic properties with temperature. If the vibrational excitation and chemical reactions takes place very rapidly in comparison to time taken by the fluid element to move in the flow field, then it is called as *equilibrium flow*. When there is sufficient time lag, then it is treated as *non-*

equilibrium flow. All these phenomena are called as *high temperature real gas effects*. The presence of high temperature reacting plasma in the vicinity of the flight vehicle influence the aerodynamic parameters, aerodynamic heating and subsequently, communication is blocked. Flight parameters like pitch, roll, drag, lift, defection of control surfaces get largely deviated from their usual estimate of calorically perfect gas. The presence of hot fluid in the vicinity of vehicle surface induces heat transfer not only through convection but also through radiation. Communication waves which are necessarily radio waves get absorbed by free electrons formed from ionization of atmospheric fluid. This phenomenon is called as *communication blackout* where on board flight parameters and ground communication is lost.

Low density flow: At standard sea level conditions, all the fluids are treated as *continuum* so that the global behavior is same as that of average fluid properties. In these conditions, the fluid contains certain desired number of molecules and the average distance between two successive collisions of the molecules is specified by its *mean free path* $\left(\lambda \approx 7\times10^{-9}\,\mathrm{m}\right)$. Since, the hypersonic flows are encountered at very high altitude (~100 km), the density of the medium is very less and the mean free path may be in the order of 0.3m. So, the air is no longer a continuous substance, rather treated as individual and widely spaced particles in the matter. Under these conditions, all the fundamental equations based on continuum assumption break down and they are dealt with the concepts of kinetic theory. This regime of the aerodynamics is known as *low-density flows*. Further increase in altitude (~ 150 km), the air density becomes so low that only a few molecules impact on the surface per unit time. This regime of flow is known as *free molecular flow*. Thus, a hypersonic vehicle moves in different flow regimes during the course of its flight i.e. from a dense atmosphere to a rarefied atmosphere. The similarity parameter that governs different regimes of the flow for certain characteristic dimension L, is then defined as Knudsen number $\left(\mathrm{Kn}\right)$.

$$\mathrm{Kn} = \frac{\lambda}{L} \tag{4.6.4}$$

Large value of Kn implies free molecular flow $(\mathrm{Kn} \to \infty)$ while small value of Kn is the regime of continuum flow $(\mathrm{Kn} < 0.2)$ as shown in Fig. 4.6.1(e). In the inviscid limit, the value of Kn approaches to zero while the free molecular flow regime begins with $\mathrm{Kn} = 1$. In the low density regimes, the *Boltzmann equation* is used to deal with the fundamental laws.

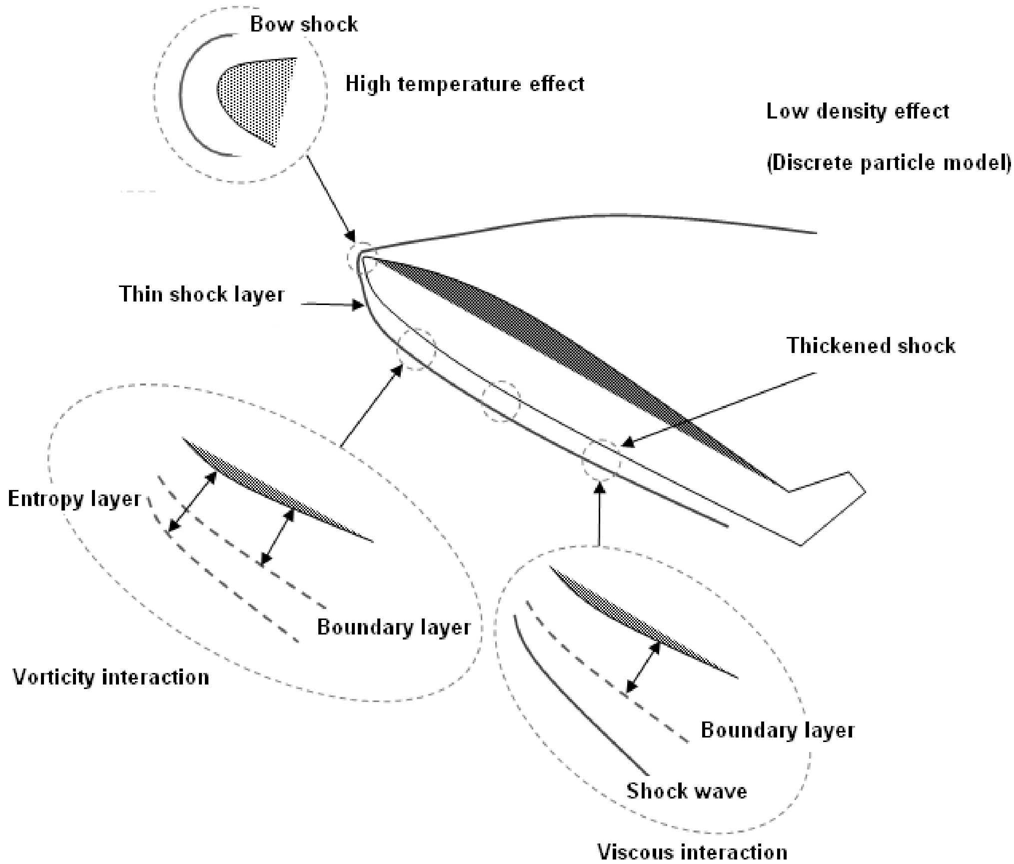

Fig. 4.6.2: Characteristics features of hypersonic flow.

From these characteristics of hypersonic flows, it is clear that Mach number to be greater than 5 is the most formal definition of hypersonic flow rather it is desired to have some of the characteristics features summarized in Fig. 4.6.2. It is more important that one of these characteristics features should appear in the flow phenomena so that the definition becomes more appropriate. There are many challenges for experimental simulation of hypersonic flow in the laboratory. Understanding the challenges faced by hypersonic flight and driving solutions these problems on case to case basic are the most research themes on hypersonic flows.

Module 4 : Lecture 7
COMPRESSIBLE FLOWS
(Hypersonic Flow: Part - II)

Inviscid Hypersonic Flow Relations

In general, the hypersonic flows are characterized with viscous boundary layers interacting the thin shock layers and entropy layers. The analysis of such flow fields is very complex flows and there are no standard solutions. In order to get some quantitative estimates, the flow field at very high Mach numbers is generally analyzed with inviscid assumption so that the mathematical complications are simplified. In conventional supersonic flows, the shock waves are usually treated as mathematical and physical discontinuities. At hypersonic speeds, some approximate forms of shock and expansion relations are obtained in the limit of high Mach numbers.

Hypersonic shock relations

Consider the flow through a straight oblique shock as shown in Fig. 4.7.1(a). The notations have their usual meaning and upstream and downstream conditions are denoted by subscripts '1' and '2', respectively. Let us revisit the exact oblique shock relations and simplify them in the limit of high Mach numbers.

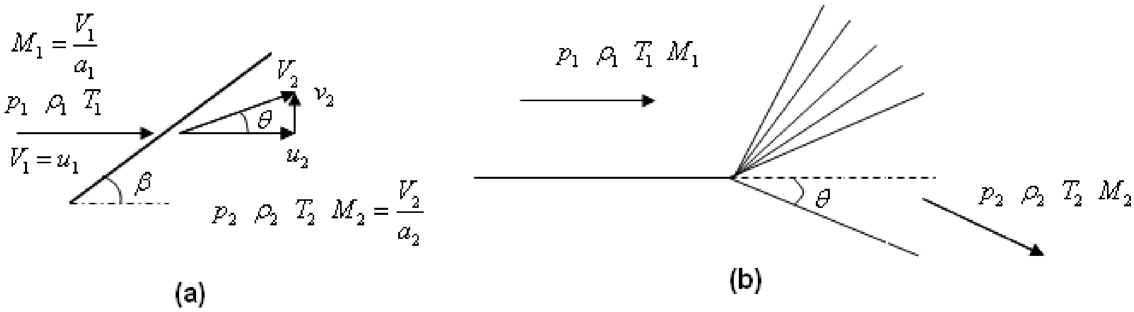

Fig. 4.7.1: Geometry of shock and expansion wave: (a) oblique shock; (b) centered expansion wave.

The exact oblique shock relations for pressure, temperature and density ratio across the wave are given by,

$$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta-1\right);\quad \frac{\rho_2}{\rho_1}=\frac{(\gamma+1)M_1^2\sin^2\beta}{2+(\gamma-1)M_1^2\sin^2\beta};\quad \frac{T_2}{T_1}=\frac{(p_2/p_1)}{(\rho_2/\rho_1)} \tag{4.7.1}$$

As, $M_1\to\infty \quad \Rightarrow M_1^2\sin^2\beta \gg 1$, so that Eq. (4.7.1) becomes,

$$\frac{p_2}{p_1}=\frac{2\gamma}{\gamma+1}M_1^2\sin^2\beta;\quad \frac{\rho_2}{\rho_1}=\frac{\gamma+1}{\gamma-1};\quad \frac{T_2}{T_1}=\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}M_1^2\sin^2\beta \tag{4.7.2}$$

It may be noted that for air $(\gamma=1.4)$ flow in the hypersonic speed limit, the density ratio approaches to a fixed value of 6. The velocity components behind the shock wave, parallel and perpendicular to the upstream flow, may be computed from the following relations;

$$\frac{u_2}{V_1}=1-\frac{2\left(M_1^2\sin^2\beta-1\right)}{(\gamma+1)M_1^2};\quad \frac{v_2}{V_1}=\frac{2\left(M_1^2\sin^2\beta-1\right)\cot\beta}{(\gamma+1)M_1^2} \tag{4.7.3}$$

For large values of M_1, the Eq. (4.7.3) can be approximated by the following relations;

$$\frac{u_2}{V_1}=1-\frac{2\sin^2\beta}{\gamma+1};\quad \frac{v_2}{V_1}=\frac{2\sin\beta\cos\beta}{(\gamma+1)}=\frac{\sin 2\beta}{\gamma+1} \tag{4.7.4}$$

The non-dimensional parameter c_p is defined as the pressure coefficient which is the ratio of static pressure difference across the shock to the dynamic pressure (q_1).

$$c_p=\frac{p_2-p_1}{q_1} \tag{4.7.5}$$

The dynamic pressure can also be expressed in the form of Mach number as given below;

$$q_1=\frac{1}{2}\rho_1V_1^2=\frac{1}{2}V_1^2\frac{\gamma p_1}{(\gamma p_1/\rho_1)}=\frac{\gamma p_1}{2}\left(\frac{V_1}{a_1}\right)^2=\frac{\gamma}{2}p_1M_1^2 \tag{4.7.6}$$

Now, Eq. (4.7.5) can be simplified as,

$$c_p=\frac{2}{\gamma M_1^2}\left(\frac{p_2}{p_1}-1\right)=\frac{4}{\gamma+1}\left(\sin^2\beta-\frac{1}{M_1^2}\right) \tag{4.7.7}$$

In the hypersonic limit of $M_1 \to \infty$, , Eq. (4.7.7) is approximated as below;

$$c_p = \left(\frac{4}{\gamma+1}\right)\sin^2\beta \qquad (4.7.8)$$

The relationship between Mach number (M), shock angle (β) and deflection angle (θ) is expressed by $\theta-\beta-M$ equation.

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta-1}{M_1^2\left(\gamma+\cos 2\beta\right)+2}\right] \qquad (4.7.9)$$

In the hypersonic limit, when, θ is small, β is also small. Thus, the small angle approximation can be used for Eq. (4.7.9).

$$\sin\beta\approx\beta;\quad \cos 2\beta\approx 1;\quad \tan\theta\approx\sin\theta\approx\theta \qquad (4.7.10)$$

It leads to simplification of Eq. (4.7.9) as below;

$$\theta = \frac{2}{\beta}\left[\frac{M_1^2\beta^2-1}{M_1^2\left(\gamma+1\right)+2}\right] \qquad (4.7.\ 11)$$

In the high Mach number limit, Eq (4.7.11) may be approximated for $\gamma = 1.4$.

$$\theta = \frac{2}{\beta}\left[\frac{M_1^2\beta^2}{M_1^2\left(\gamma+1\right)}\right] = \frac{2\beta}{\gamma+1};\quad \frac{\beta}{\theta} = \frac{\gamma+1}{2} \text{ and } \beta = 1.2\theta \qquad (4.7.\ 12)$$

It is interesting to observe that in the hypersonic limit of a slender wedge, the shock wave angle is only 20% larger than the wedge angle which is the typical physical features of thin shock layer in the hypersonic flow.

Hypersonic expansion wave relations

Consider the flow through an expansion corner as shown in Fig. 4.7.1(b). The expansion fan consists of infinite number of Mach waves originating at the corner and spreading downstream. The notations have their usual meaning and upstream and downstream conditions are denoted by subscripts '1' and '2', respectively. Let us revisit the exact relations for a *Prandtl-Meyer* expansion. The relation for deflection angle θ, M_1 and M_2 is expressed through *Prandtl-Meyer* function $\{\nu(M)\}$.

$$\nu(M)=\sqrt{\frac{\gamma+1}{\gamma-1}}\left[\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right]-\tan^{-1}\sqrt{M^2-1};\quad \theta=\nu(M_2)-\nu(M_1) \tag{4.7. 13}$$

For large Mach numbers, $\sqrt{M_1^2-1}\approx M$ and series expansion can be approximated for the trigonometric functions.

$$\begin{aligned}\nu(M)&=\left(\sqrt{\frac{\gamma+1}{\gamma-1}}\right)\left(\frac{\pi}{2}\right)-\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{1}{M}\right)-\frac{\pi}{2}+\frac{1}{M}\\ \text{and}\quad \theta&=\frac{1}{M_2}-\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{1}{M_2}\right)-\frac{1}{M_1}+\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{1}{M_1}\right)\end{aligned} \tag{4.7. 14}$$

Further, simplification of Eq (4.7.14) can be done and the final expression for θ may be written as below;

$$\theta=\frac{2}{\gamma-1}\left(\frac{1}{M_1}-\frac{1}{M_2}\right) \tag{4.7. 15}$$

Hypersonic Similarity Parameter

In the study of hypersonic flow over slender bodies, the product of $M_1\theta$ is a controlling parameter which is known as the similarity parameter denoted by K. All the hypersonic shock and expansion relations can be expressed in terms of this parameter. Introducing this parameter, Eq. (4.7.11) is rewritten in the limit of high values of Mach number;

$$M_1^2\beta^2-1=\left[\frac{M_1^2(\gamma+1)}{2}+1\right]\beta\theta \Rightarrow M_1^2\beta^2-1=\left(\frac{\gamma+1}{2}\right)M_1^2\beta\theta \tag{4.7.16}$$

Rearranging Eq. (4.7.16), one may obtain a quadratic equation in terms of (β/θ), which may be easily solved.

$$\left(\frac{\beta}{\theta}\right)^2-\frac{\gamma+1}{2}\left(\frac{\beta}{\theta}\right)-\frac{1}{M_1^2\theta^2}=0 \Rightarrow \frac{\beta}{\theta}=\frac{\gamma+1}{4}+\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{M_1^2\theta^2}} \qquad (4.7.\ 17)$$

Within the framework of hypersonic assumption, the hypersonic shock relation for pressure ratio (Eq. (4.7.1), may be reduced in terms of K by using Eq. (4.7.17).

$$\frac{p_2}{p_1}=1+\frac{\gamma(\gamma+1)}{4}K^2+\gamma K^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{K^2}} \qquad (4.7.18)$$

Similarly, the pressure coefficient may also be expressed as a function of similarity parameter.

$$c_p=2\theta^2\left[\frac{\gamma+1}{4}+\sqrt{\left(\frac{\gamma+1}{4}\right)^2+\frac{1}{K^2}}\right] \Rightarrow \frac{c_p}{\theta^2}=f(K,\gamma) \qquad (4.7.19)$$

The similarity relations for *Prandtl-Meyer expansion wave* may also be written in terms of the similarity parameter. The flow through an expansion fan is isentropic. Hence, the isentropic relations for pressure can be used for the conditions on both sides of expansion fan. When approximated to hypersonic flows, the static pressure relation across the expansion fan can be written as below;

$$\frac{p_2}{p_1}=\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)M_2^2}\right]^{\frac{\gamma}{\gamma-1}} \Rightarrow \frac{p_2}{p_1}=\left(\frac{M_1}{M_2}\right)^{\frac{2\gamma}{\gamma-1}} \qquad (4.7.20)$$

Rearranging Eq. (4.7.15), the ratio of Mach numbers across the expansion wave can be obtained.

$$\frac{M_1}{M_2}=1-\left(\frac{\gamma-1}{2}\right)M_1\theta \qquad (4.7.\ 21)$$

Combine Eqs. (4.7.20 & 4.7.21) to obtain pressure ratio across the expansion fan in terms of similarity parameter.

$$\frac{p_2}{p_1}=\left(1-\frac{\gamma-1}{2}K\right)^{\frac{2\gamma}{\gamma-1}} \qquad (4.7.22)$$

Further, the pressure coefficient across the expansion fan, may be expressed as a function of similarity parameter.

$$c_p = \frac{2}{\gamma M_1^2}\left(\frac{p_2}{p_1}-1\right) = \frac{2}{\gamma M_1^2}\left[\left(1-\frac{\gamma-1}{2}K\right)^{\frac{2\gamma}{\gamma-1}}-1\right] \tag{4.7.23}$$

Multiply and divide the right-hand side by θ^2 and simplify to obtain the following relation.

$$c_p = \frac{2\theta^2}{\gamma K^2}\left[\left(1-\frac{\gamma-1}{2}K\right)^{\frac{2\gamma}{\gamma-1}}-1\right] \Rightarrow \frac{c_p}{\theta^2} = g\left(K,\gamma\right) \tag{4.7.24}$$

It may be seen that pressure coefficient for hypersonic shock and expansion wave, are related through the similarity parameter in the limit of hypersonic Mach numbers. Hence, the Eqs (4.7.19 & 4.7.24) are analogous.

Module 4 : Lecture 8
COMPRESSIBLE FLOWS
(Hypersonic Flow: Part - III)

Newtonian Theory for Hypersonic Flows

The hypersonic flows are highly nonlinear due to many physical phenomena leading to complexity in the mathematical formulation and its solution. One can get rid of the complex nature of aerodynamic theories with the simple approximation of inviscid flow to obtain the linear relationship. It is interesting to note that the invicid compressible flow theory for high Mach number flows, resemble the fundamental Newtonian law of classical mechanics.

When a fluid as a stream of particles in rectilinear motion, strikes a plate, it loses all its momentum normal to the surface and moves tangentially to the surface without the loss of tangential momentum. This is known as the Newtonian impact theory as shown in Fig. 4.8.1(a). Let a fluid stream of density ρ_∞ strikes a surface of area A, with a velocity V_∞. This surface is inclined at an angle θ with the free stream. By Newton's law, the time rate of change of momentum of this mass flux is equal to the force (F) exerted on the surface.

$$F = (\rho_\infty A)(V_\infty \sin\theta)(V_\infty \sin\theta) = \rho_\infty V_\infty^2 A \sin^2\theta \quad \Rightarrow \frac{F}{A} = \rho_\infty V_\infty^2 \sin^2\theta \qquad (4.8.1)$$

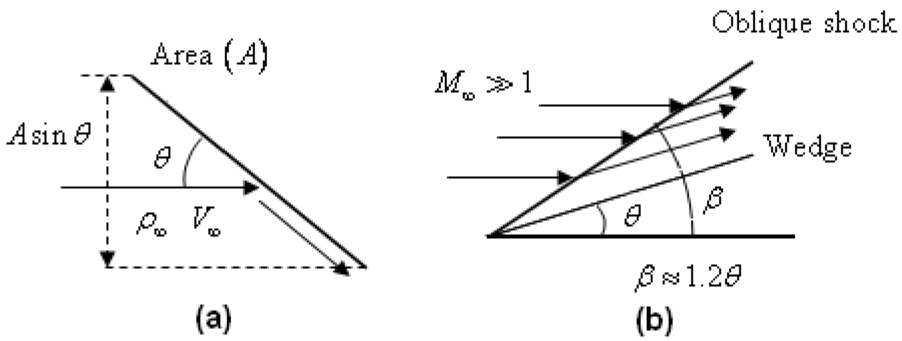

Fig. 4.8.1: Newtonian impact theory and hypersonic flow over a wedge: (a) schematic representation of a jet striking a plate; (b) streamlines in a thin shock layer.

Since the motion is rectilinear and the individual particles do not interact with each other, the force per unit area, associated with the random motion may be interpreted as the difference in surface pressure (p) and the free stream pressure (p_∞). So, the Eq. (4.8.1) may be simplified in terms of pressure coefficient (c_p).

$$c_p = \frac{p - p_\infty}{(1/2)\rho_\infty V_\infty^2} = 2\sin^2\theta \qquad (4.8.2)$$

Now, let us analyze the hypersonic flow over a wedge with inclination angle θ as shown in Fig. 4.8.1(b). Both the upstream and downstream side of the shock wave, the streamlines are straight and parallel. But, the stream lines are deflected by an angle θ in the downstream. Since, the difference in the shock wave angle (β) and the flow deflection is very small at hypersonic speeds, it may be visualized as the upstream incoming flow impinging on the wedge surface and then running parallel to the wedge surface in the downstream. This phenomenon is analogous to Newtonian theory and Eq. (4.8.2) may be used for hypersonic flow as well to predict the surface pressures. It is known as the *Newtonian Sine-Squared Law* for hypersonic flow.

Inviscid Hypersonic Flow over a Flat Plate

Consider a two-dimensional flat plate of certain length (l), inclined at angle (θ) with respect to free stream hypersonic flow (Fig. 4.8.2). Now, the Newtonian theory can be applied at the lower and upper surface of the plate to obtain the pressure coefficient (c_p).

$$c_{pl} = 2\sin^2\theta;\ c_{pu} = 0 \qquad (4.8.3)$$

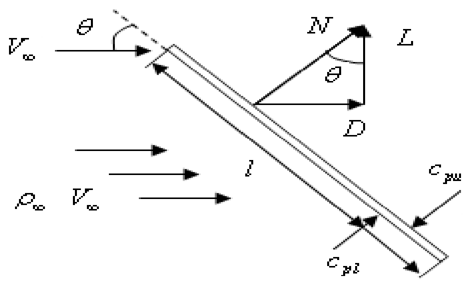

Fig. 4.8.2: Illustration of aerodynamic forces for a flat plate in hypersonic flow.

The difference in pressures in the upper and lower surface of the plate, gives rise to a normal force (N). The normal force coefficient (c_n) can also be readily defined through the following formula.

$$c_n = \frac{1}{l}\int_0^l \left(c_{pl} - c_{pu}\right) dx = \frac{N}{q_\infty S} \tag{4.8.4}$$

Here, $q_\infty \left(= \frac{1}{2}\rho_\infty V_\infty^2\right)$ is the free stream dynamic pressure, $S(=l)$ is the frontal area per unit width and x is the distance along the length of the plate from the leading edge. Now, substitute Eq. (4.8.3) in Eq. (4.8.4) to obtain the simplified relations;

$$c_n = \frac{1}{l}\left(2\sin^2\theta\right)l = 2\sin^2\theta \tag{4.8.5}$$

If L and D are defined as the lift and drag as shown in Fig. 4.8.2, then the other aerodynamic parameters such as lift coefficient (c_l) and drag coefficient (c_d) can be expressed in the following fashion.

$$c_l = \frac{L}{q_\infty S} = c_n \cos\theta = 2\sin^2\theta\cos\theta; \;\; c_d = \frac{D}{q_\infty S} = c_d \cos\theta = 2\sin^3\theta \tag{4.8.6}$$

Referring to geometry of Fig. 4.8.2, the other important parameter *lift-to-drag* is obtained through the following relation;

$$\frac{L}{D} = \cot\theta \tag{4.8.7}$$

The results of Newtonian theory for the inviscid flow over a flat plate are plotted in Fig. 4.8.3 and the following important observations can be made;

- The value of *lift-to-drag* ratio increases monotonically when the inclination angle decreases. It is mainly due to the fact that the Newtonian theory does not account for skin friction drag in the calculation. When skin friction is added, the drag becomes a finite value at 0^0 inclination angle and the ratio approaches zero.
- The lift curve reaches its peak value approximately at an angle of 55^0. It is quite realistic, because most of the practical hypersonic vehicles get their maximum lift in this vicinity of angle of attack.
- The lift curve at lower angle ($0\text{-}15^0$) shows the non-linear behavior. It is clearly the important characteristics feature of the hypersonic flows.

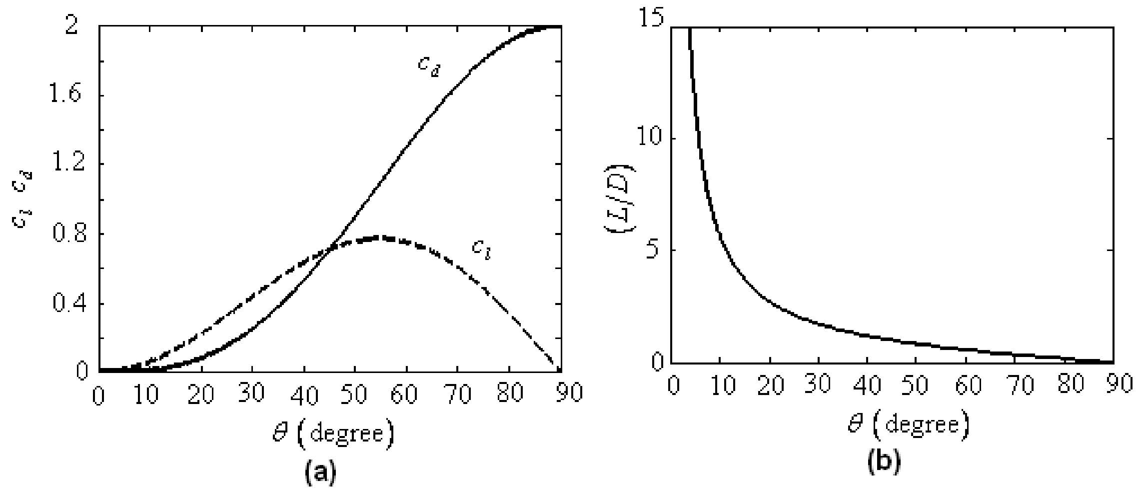

Fig. 4.8.3: Aerodynamic parameters for a flat plate inclined at an angle.

Mach number Independence Principle

Precisely, this principle states that certain aerodynamic quantities, such as pressure coefficient, lift and wave drag coefficients and flow-field structure (shock wave shapes and Mach wave patterns), become relatively independent on Mach number when its value is made sufficiently large. Let us justify this principle based on the following analysis;

<u>Oblique Shock Relations</u>: Let us revisit the following oblique shock relations when approximated for hypersonic Mach numbers;

$$\begin{aligned} &\frac{u_2}{V_1} = 1 - \frac{2\sin^2\beta}{\gamma+1}; \quad \frac{v_2}{V_1} = \frac{2\sin\beta\cos\beta}{(\gamma+1)} = \frac{\sin 2\beta}{\gamma+1} \\ &c_p = \left(\frac{4}{\gamma+1}\right)\sin^2\beta; \quad \frac{\beta}{\theta} = \frac{\gamma+1}{2} \end{aligned} \tag{4.8.8}$$

It may be observed here that the oblique shock relations turn down to simplified form in the regime of hypersonic Mach numbers. Eq. (4.8.8) does not bear the Mach number term and thus the flow field is also independent of Mach number. This is called as *Mach number independence principle* and valid for very high Mach number inviscid flows.

<u>Newtonian Theory</u>: The interesting feature of hypersonic flows, is the fact that certain aerodynamic parameters calculated from Newtonian theory, do not explicitly depend on the Mach number. Of course, these equations implicitly assume that the Mach numbers are high enough for hypersonic flows to prevail but its precise value do not enter into the calculations. In fact, the pressure and force coefficients expressed in Eqs (4.8.2- 4.8.7) do not contain the Mach number term. When extended to cylinder and sphere, the Newtonian theory predicts the drag coefficient of values as 1.33 and 1, respectively, irrespective of Mach number. This particular feature of hypersonic flow is known as *Mach number independence* and the Newtonian results are the consequence of this principle.

Modified Newtonian Theory

In order to predict the pressure distributions $\left(c_p\right)$ over blunt shaped aerodynamic bodies, the Newtonian theory (Eq. 4.8.2) is modified by the following expression.

$$c_p = c_{p\max} \sin^2\theta; \quad c_{p\max} = \frac{p_{02} - p_\infty}{\left(1/2\right)\rho_\infty V_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_{02}}{p_\infty} - 1\right) \tag{4.8.9}$$

Here, $c_{p\max}$ is the maximum value of pressure coefficient, evaluated at stagnation point behind the normal shock, $p_\infty, \rho_\infty, M_\infty$ are the free stream values of static pressure, static density, Mach number, respectively and p_{02} is the stagnation pressure behind the normal shock. From the normal shock relations, it is possible to obtain the pressure ratio appearing in Eq. (4.8.9) for calculation of $c_{p\max}$.

$$\frac{p_{02}}{p_\infty} = \left[\frac{\left(\gamma+1\right)^2 M_\infty^2}{4\gamma M_\infty^2 - 2\left(\gamma-1\right)}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{1-\gamma\left(1-2M_\infty^2\right)}{\gamma+1}\right] \tag{4.8.10}$$

Substitute Eq. (4.8.10) in Eq. (4.8.9) to obtain $c_{p\max}$.

$$c_{p\max} = \frac{2}{\gamma M_\infty^2}\left\{\left[\frac{\left(\gamma+1\right)^2 M_\infty^2}{4\gamma M_\infty^2 - 2\left(\gamma-1\right)}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{1-\gamma\left(1-2M_\infty^2\right)}{\gamma+1}\right] - 1\right\} \tag{4.8.11}$$

The relation of $c_{p\max}$ as a function of free stream Mach number and specific heat ration for the gas is plotted in Fig. 4.8.4.

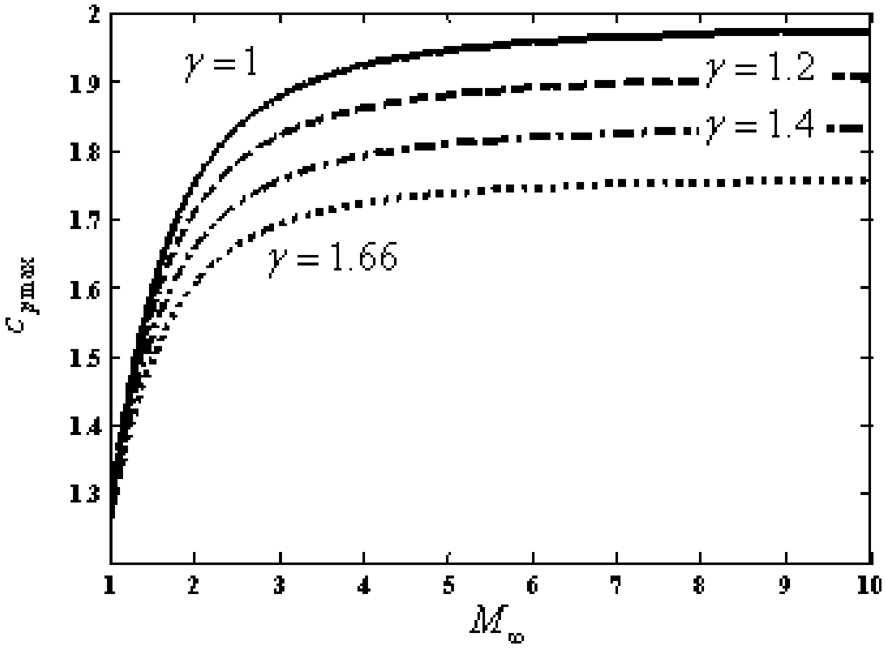

Fig. 4.8.4: Variation of stagnation pressure coefficient as a function of free stream Mach number and specific heat ratio.

In the limit of $M_\infty \to \infty$, $c_{p\max}$ can be obtained as below;

$$\begin{aligned} c_{p\max} &\to \left[\frac{(\gamma+1)^2}{4\gamma}\right]^{\frac{\gamma}{\gamma-1}} \left(\frac{4}{\gamma+1}\right) \\ &\to 1.839 \;\; (\gamma = 1.4) \\ &\to 2 \;\; (\gamma = 1) \end{aligned} \tag{4.8.12}$$

The Eq. (4.8.9) with the $c_{p\max}$ given by the expression in Eq. (4.8.12) is called as the *modified Newtonian law*. The following important observation may be made.

- The modified Newtonian law does not follow the Mach number independence principle.
- When both $M_\infty \to \infty$ and $\gamma \to 1$, the straight Newtonian law is recovered from modified theory.
- The modified Newtonian theory is a very important tool to estimate the pressure coefficients in the stagnation regions in the hypersonic flow fields of the blunt bodies.