ME181605: Heat Transfer - II

Module 1: Fundamentals of Convective Heat Transfer

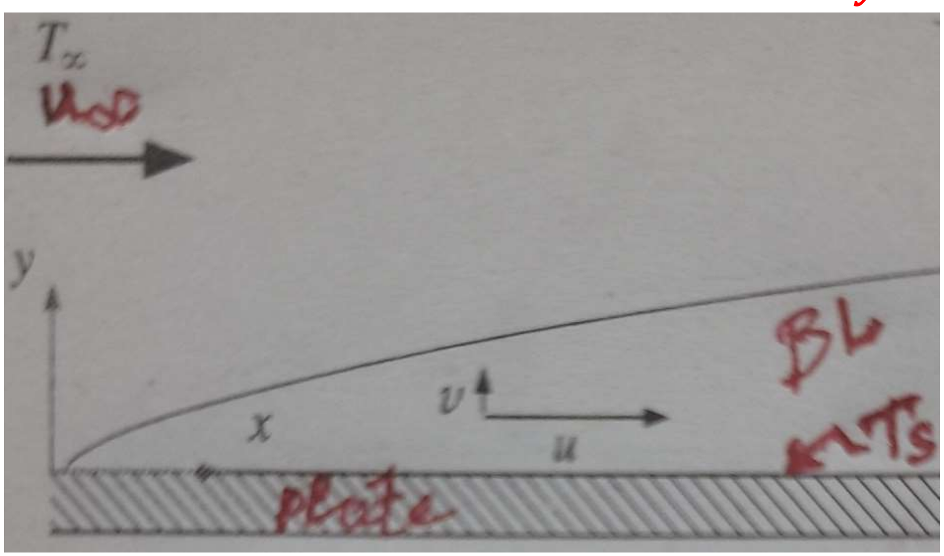
Department of Mechanical Engineering

Lecture by Dr. Mantulal Basumatary

Assistant Professor

Forced convection over a flat plate (Cont.)

- Boundary layer approximation: Inside the boundary layer of laminar flow the following conditions exist when we neglect the body forces like gravity and magnetic force:
  - Velocity components: $\boldsymbol{u \gg v}$
  - Velocity gradients: $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial y} \ll \dfrac{\partial u}{\partial y}$
  - Temperature gradients: $\dfrac{\partial T}{\partial x} \ll \dfrac{\partial T}{\partial y}$



These simplifications are known as the *boundary layer approximation.*

If we apply the boundary layer approximation in the *y*-momentum equation (derivation of momentum will be discussed in the coming slides), we will end up to $\partial P/\partial y = 0$ (i.e. variation of pressure occurs along the *x*-axis only), where P denotes the pressure inside the BL. Thus, $P = P(x)$ and therefore, inside the boundary layer, $\dfrac{\partial P}{\partial x} = \dfrac{dP}{dx}$

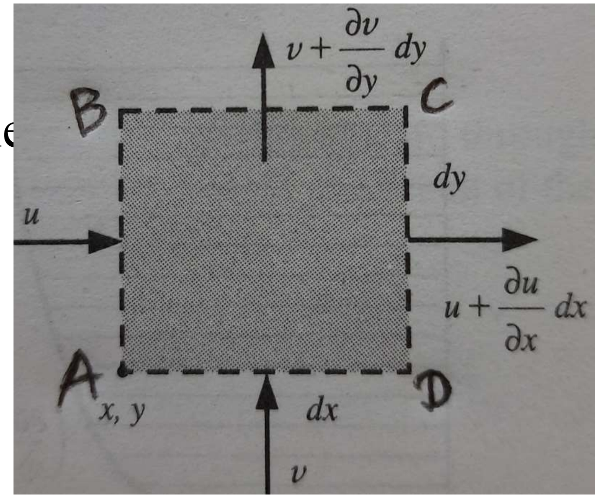
❖ Governing equations involved in the convective heat transfer

We have already understood that convection is the mode of heat transfer involving conduction and fluid motion. Therefore, studies of convective heat transfer problems require the detailed knowledge about the equations that govern the conduction and fluid motion.

- Fluid motion is governed by the (a) continuity and (b) momentum equations.
- Conduction is governed by the (c) energy equation.

a) Continuity Equation: This equation is derived from the principle of conservation of mass. It states that *mass cannot be created or destroyed during a process.*

Let us consider a two-dimensional differential control volume having dimension **$dx \times dy \times 1 = dx \times dy$ ($z = 1$ unit)** as shown in the



And, u and v denote the x and y component of velocities at the inlet.

Therefore, the corresponding velocities at the outlet are $\left(u + \frac{\partial u}{\partial x} dx\right)$ and $\left(v + \frac{\partial v}{\partial y} dy\right)$.

Governing equations involved in the convective heat transfer (Cont.)

In a steady flow, the amount of mass within the control volume remains constant, and thus the conservation of mass can be expressed as

(Rate of mass flow into the control volume) = (Rate of mass flow out of the control volume)

We know,

Mass flow rate = density × average velocity × cross sectional area normal to flow

Mass flow rate at the inlets AB and AD $= \rho u(dy.1) + \rho v(dx.1) = \rho u dy + \rho v dx$

And,

Mass flow rate at the outlets CD and BC $= \rho(u + \frac{\partial u}{\partial x} dx)dy + \rho(v + \frac{\partial v}{\partial y} dy)dx$

Substituting in the above mass conservation equation, we obtain

$$\rho u dy + \rho v dx = \rho(u + \frac{\partial u}{\partial x} dx)dy + \rho(v + \frac{\partial v}{\partial y} dy)dx$$

$$= \rho u dy + \rho \frac{\partial u}{\partial x} dx dy + \rho v dx + \rho \frac{\partial v}{\partial y} dy dx$$

Governing equations involved in the convective heat transfer (Cont.)

After simplification and dividing both sides by *dxdy*, yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

This is the equation of conservation of mass in differential form and is popularly known as the ***continuity equation*** for steady two-dimensional incompressible (constant density) flow.

Remark: The above continuity equation is valid for the two-dimensional flow in the Cartesian coordinate system.

Governing equations involved in the convective heat transfer (Cont.)

b) Momentum equations: derived from the Newton's second law of motion. It states that *the net force acting on a control volume is equal to the mass times the acceleration of the fluid element within the control volume.*

- Forces acting on the control volume consists of-
  - Body forces: They act throughout the entire body of the control volume. For example, gravity, electric and magnetic forces.
  - Surface forces: They act on the control surface and are proportional to the surface area. For example, pressure force due to hydrostatic pressure and shear stresses due to viscous effects.

We can express the Newton's second law of motion for the control volume as

(Mass)×(Acceleration in the specified direction) = (Net force acting in that direction)

$$\delta m \times a_x = F_{surface,x} + F_{body,x}$$

Note: Pressure represents the compressive force applied on the fluid element by the surrounding fluid, and is always directed to the surface. Shear force acts along /tangential to the control surface.

where, $\delta m = \rho dxdy$, the mass of the fluid element within the control volume.

We have already assumed that the flow is steady and two-dimensional and thus $u = u(x,y)$. Therefore, the total differential of u is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

We know, the acceleration of the fluid element in the x direction is

$$a_x = \frac{du}{dt}$$

Now, substituting the value of du in the above equation, we get

$$a_x = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Since, $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$ $\quad\quad \therefore \; a_x = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v$

Rearranging the above equation,

Governing equations involved in the convective heat transfer (Cont.)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

The above expression of acceleration seems to be unintuitive because it does not contain derivative of velocity with respect to time and more interestingly it is a steady flow. If we account for the unsteady flow, the above equation becomes as

$$a_x = \underbrace{\frac{\partial u}{\partial t}}_{\substack{\text{Local Acceleration} \\ \text{or} \\ \text{Temporal Acceleration}}} + \underbrace{\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v}_{\substack{\text{Convective Acceleration} \\ \text{or} \\ \text{Advective Acceleration}}}$$

Governing equations involved in the convective heat transfer (Cont.)

- Local acceleration: It is the change of velocity at a specified location with respect to time.
- Convective acceleration: It is the change of velocity from one position to another as the fluid flows and therefore it exists even in steady flow. We can easily observe it by putting a paper boat in a channel where it has converging area of flow when the flow is steady.

The net surface force acting in the *x*-direction becomes

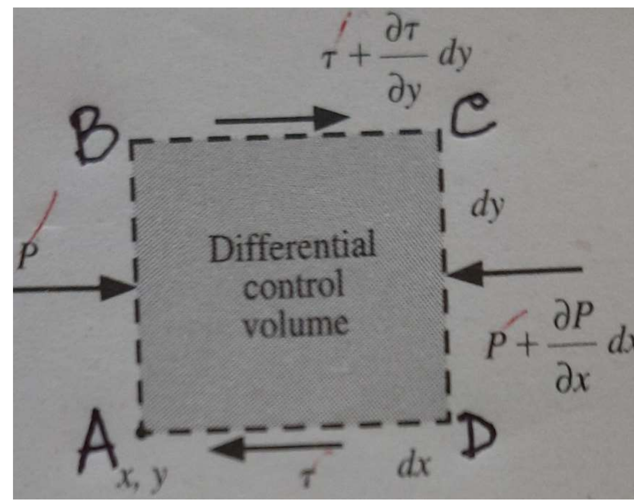
$$F_{surface,x} = \left(\frac{\partial \tau}{\partial y} dy\right)(dx.1) - \left(\frac{\partial P}{\partial x} dx\right)(dy.1) = \left(\frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x}\right) dxdy$$

For Newtonian fluid, we know,

$$\tau = \mu \frac{\partial u}{\partial y}$$

Substituting the value of τ,

$$\therefore F_{surface,x} = \left(\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}\right) dxdy$$



Note: The normal stress is related to the velocity gradients $\partial u/\partial x$ and $\partial v/\partial y$, that are much smaller than $\partial u/\partial y$ to which shear stress is related. Therefore, the contribution of normal stress has been neglected.

Governing equations involved in the convective heat transfer (Cont.)

- Substituting the values in the above equation of Newton's second law of motion and after simplification, we get

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial P}{\partial x}$$

This relation is known as the *x-momentum equation* for steady flow in Cartesian coordinate system. If there is a body force acting in the *x*-direction, it can be added to the right side of the momentum equation provided that it is expressed per unit volume of the fluid.

Governing equations involved in the convective heat transfer (Cont.)

c) Energy equation: derived from the conservation of energy. It states that *energy can never be created nor destroyed during a process*. The energy balance for a steady flow process can be expressed as

$$E_{in} - E_{out} = 0$$

Then, the rate of energy balance for a steady flow process can be expressed as

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

▪ Energy may be transferred by heat, work, and mass only. Thus, we can write the above equation as follows,

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{heat} + \left(\dot{E}_{in} - \dot{E}_{out}\right)_{work} + \left(\dot{E}_{in} - \dot{E}_{out}\right)_{mass} = 0$$

The total energy of a flowing fluid stream per unit mass is given by

$$e = h + KE + PE$$

where, h, KE and PE respectively represent the enthalpy, kinetic energy and potential energy. Here, we have neglected the kinetic and potential energy because they are usually negligible relative to the enthalpy. $\therefore e = h = c_p T$, where, c_p is the specific heat of the fluid.

Governing equations involved in the convective heat transfer (Cont.)

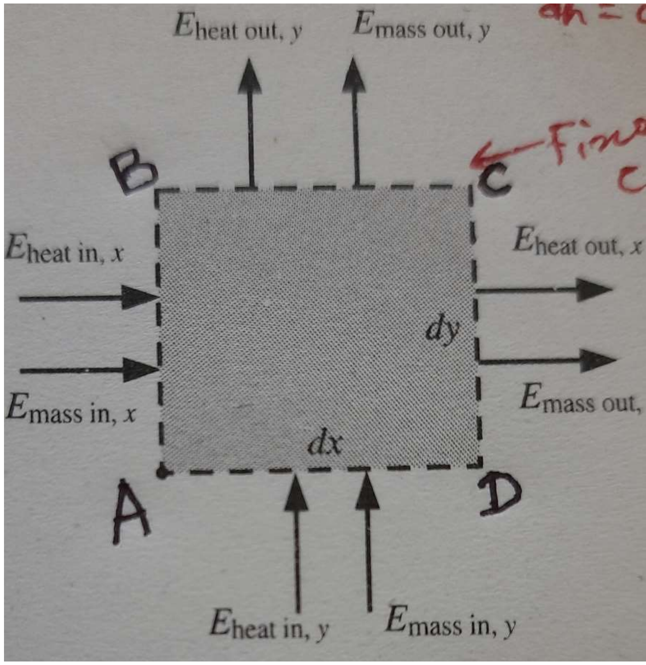
- Energy is a scalar quantity, and thus energy interactions in all directions can be combined in one equation.

The rate of energy transfer to the control volume by mass in the x-direction is

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{mass,x} = (\dot{m}e)_x - \left[(\dot{m}e)_x - \frac{\partial(\dot{m}e)_x}{\partial x}dx\right]$$

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{mass,x} = -\frac{\partial(\rho u dy c_p T)}{\partial x}dx$$

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{mass,x} = -\rho c_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dxdy$$



Similarly, repeating same step for y-direction and adding the results, the net rate of energy transfer to the control volume by mass is obtained after simplification to be

Governing equations involved in the convective heat transfer (Cont.)

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{mass} = -\rho c_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dxdy - \rho c_p\left(v\frac{\partial T}{\partial y} + T\frac{\partial v}{\partial y}\right)dxdy$$

$$= -\rho c_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x} + v\frac{\partial T}{\partial y} + T\frac{\partial v}{\partial y}\right)dxdy$$

$$= -\rho c_p\left[\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]dxdy$$

$$= -\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)dxdy$$ [$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, continuity equation]

The net rate of energy transfer due to conduction to the fluid element in the *x*-direction is

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{heat,x} = \dot{Q}_x - \left(\dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x}dx\right)$$

$$= -\frac{\partial}{\partial x}\left(-kdy\frac{\partial T}{\partial x}\right)dx$$

$$= k\frac{\partial^2 T}{\partial x^2}dxdy$$

Governing equations involved in the convective heat transfer (Cont.)

Similarly, repeating same step for *y*-direction and adding the results, the net rate of energy transfer to the control volume by conduction is obtained after simplification to be

$$\left(\dot{E}_{in} - \dot{E}_{out}\right)_{heat,x} = k\frac{\partial^2 T}{\partial x^2}dxdy + k\frac{\partial^2 T}{\partial y^2}dxdy$$

$$= k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)dxdy$$

- The work done by the body forces is determined by multiplying it with the velocity in the direction of the force and the volume of fluid element. This work needs to be considered only in the presence of significant gravitational, electric and magnetic effects. The energy transfer contributed from work done by the shear forces are negligible relative to the other mode of energy transfer.

Therefore, the energy equation for the steady two-dimensional flow of a fluid with constant properties and negligible shear stresses is obtained substituting the value of above expressions in the energy equation and after simplification, give

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Governing equations involved in the convective heat transfer (Cont.)

When the viscous shear stresses are not negligible, their effect is accounted for by expressing the energy equation as

$$\rho c p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi$$

where, Φ denotes the viscous dissipation function and is obtained from the following expression,

$$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

Thank You