Module-1
Viscous Flow

- Derivation and solution to **Navier-Stokes equation.**

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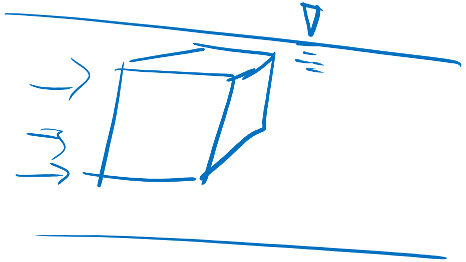
Navier-Stoke's eq^n :

Newton's 2nd Law of motion,

$$F_x = m a_x$$



For fluid flow,

$$F_x = F_B + F_P + F_V + \boxed{F_T + F_C}$$

In case of viscous flow,

$$F_x = \boxed{\underbrace{F_B}_{\text{Need not to calculate}} + \underbrace{F_P}_{\sigma} + \underbrace{F_V}_{\tau} = m \cdot a_x}$$

$$\frac{* \; a_y}{* \; a_z}$$

Acceleration in 3D

$$V = u\hat{i} + v\hat{j} + w\hat{k}$$

$$u = f(t, x, y, z), \quad v = f(t, x, y, z), \quad w = f(t, x, y, z)$$

$$du = \frac{\partial u}{\partial t} \cdot dt + \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz$$

$$\frac{dU}{dt} = \frac{du}{dt} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$\checkmark \; a_x = \frac{dU}{dT} = \frac{du}{dt} + u \cdot \frac{du}{dx} + v \cdot \frac{du}{dy} + w \cdot \frac{du}{dz}$$

$$\checkmark \; a_y = \frac{dV}{dT} = \frac{dV}{dt} + u \cdot \frac{dv}{dx} + V \cdot \frac{dv}{dy} + w \cdot \frac{dv}{dz}$$

$$\checkmark \; a_z = \frac{dw}{dT} = \frac{dw}{dz} + u \cdot \frac{dw}{dx} + v \cdot \frac{dw}{dy} + w \cdot \frac{dw}{dz}$$

Total acceleration = acceleration (t) + acceleration (space)

$$\boxed{\text{Total acceleration} = \text{Local or Temporal acceleration} + \text{Convective acceleration.}}$$

$$\frac{D}{DT} = \frac{d}{dt} + u \cdot \frac{d}{dx} + v \cdot \frac{d}{dy} + w \cdot \frac{d}{dz}$$

$$\boxed{a = \boxed{a_{Local} + a_{convective}}}$$

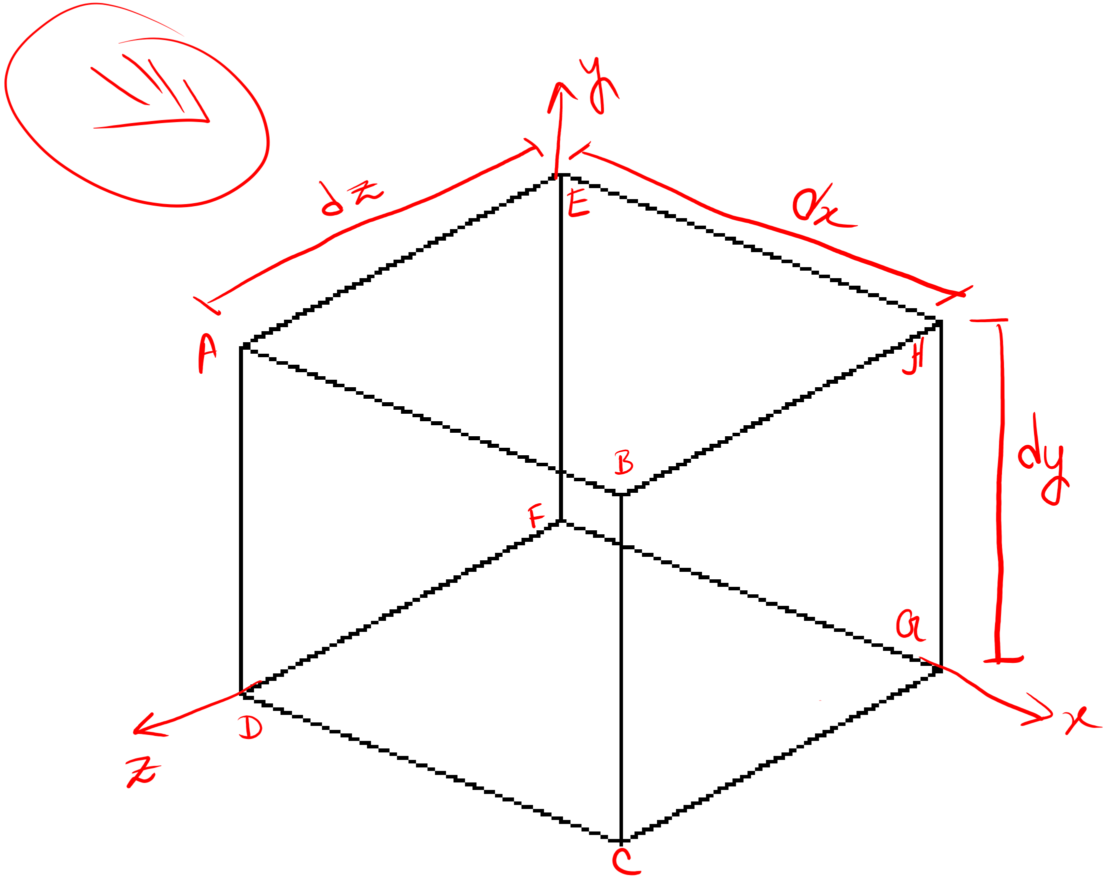
Newton's 2nd Law,

$$F_B \quad F_P + F_V = m \cdot a$$

$$F_P \rightarrow \sigma \qquad F_V \rightarrow \tau$$

Body is in fluid (3-D flow)

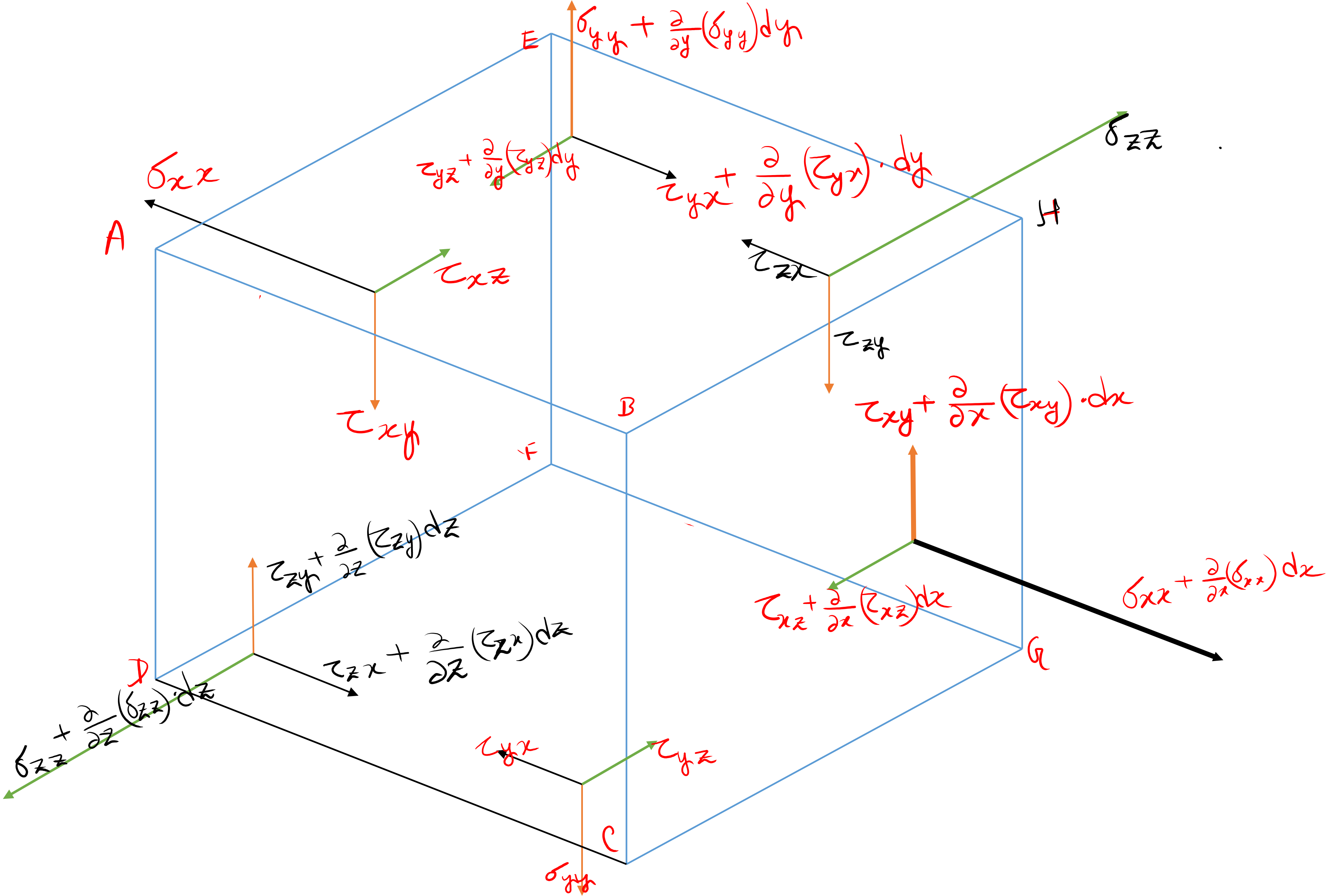
↳ Cartesian coordinate



$\boxed{A_{12}}$

1 → Plane ⊥ to which axis

2 → Acting in the direction



$F_p \rightarrow \sigma$

x direction

$$\left\{\begin{aligned} F_{p_1} &= -\sigma_{xx}\cdot dy\cdot dz \\ F_{p_2} &= \left[\sigma_{xx} + \frac{\partial}{\partial x}(\sigma_{xx})\,dx\right] dy\cdot dz \end{aligned}\right\}$$

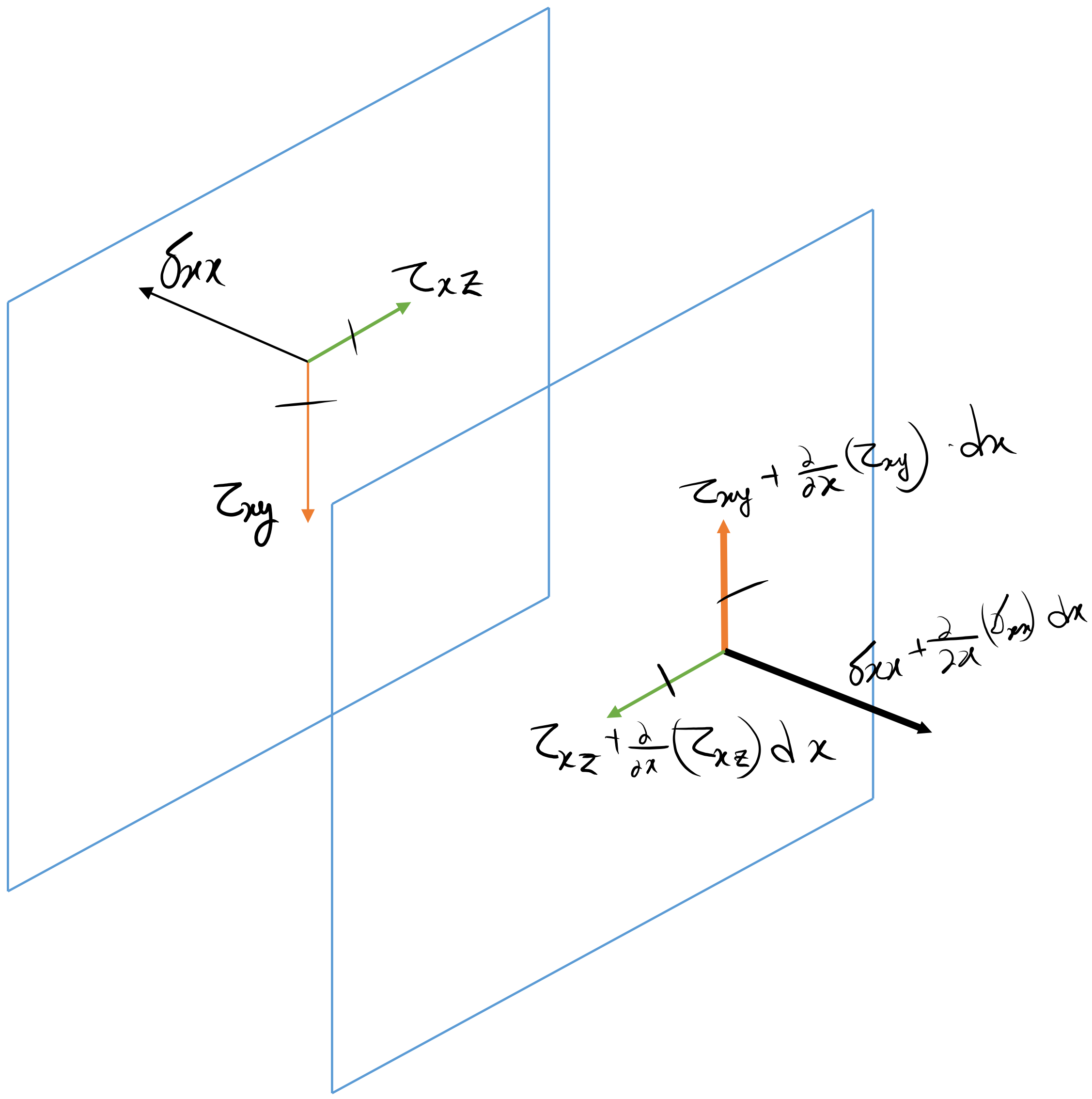
$F_V \rightarrow \tau$

z direction

$$\left\{\begin{aligned} &-\tau_{xz}\cdot dy\,dz \\ &\left[\tau_{xz} + \frac{\partial}{\partial x}(\tau_{xz})\cdot dx\right] dy\cdot dz \end{aligned}\right\}$$

y direction

$$\left\{\begin{aligned} &-\tau_{xy}\cdot dy\cdot dz \\ &+\left[\tau_{xy} + \frac{\partial}{\partial x}(\tau_{xy})\right] dy\,dz \end{aligned}\right\}$$



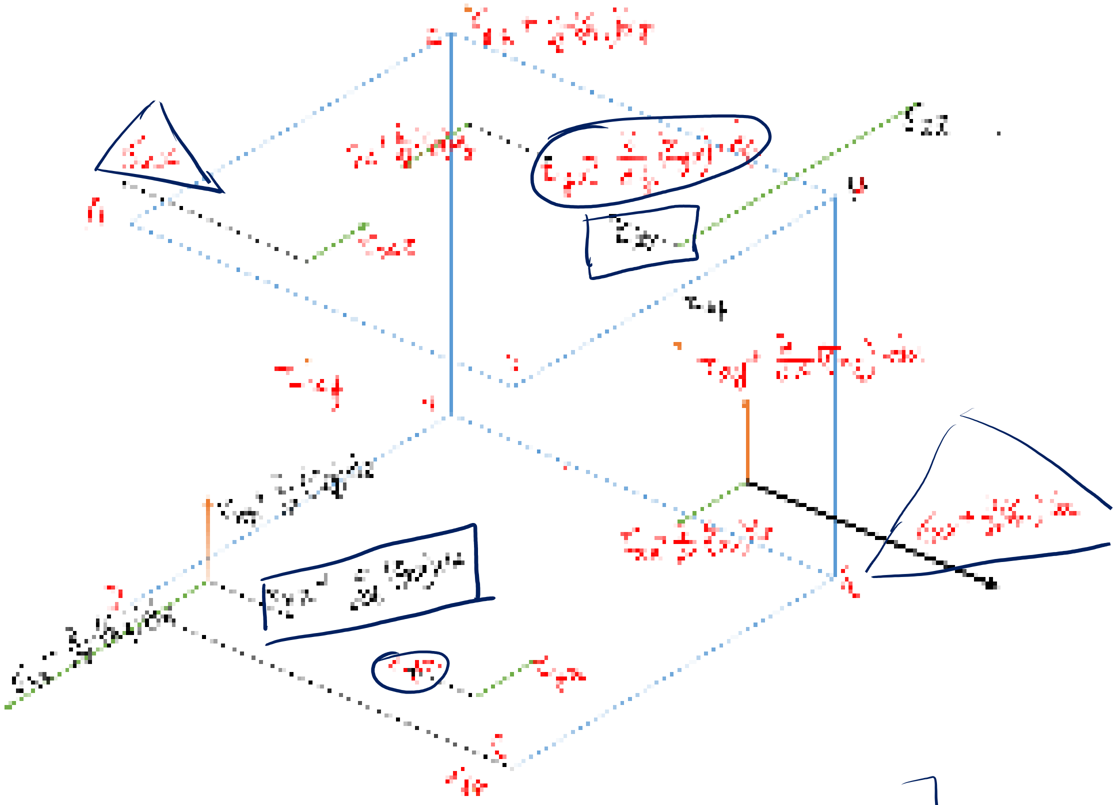
$$\left[F_{B_x} + \underline{F_{P_x} + F_{V_x}} = m \cdot a_x \right]$$

In x-direction,

$$F_{P_x} + \boxed{F_{V_x}} = -\sigma_{xx} \cdot dy\,dz + \left[\sigma_{xx} + \frac{\partial}{\partial x}(\sigma_{xx}) \cdot dx\right] \cdot dy\,dz$$

$$-\tau_{zx} \cdot dx \cdot dy + \left[\tau_{zx} + \frac{\partial}{\partial z}(\tau_{zx}) \cdot dz\right] dx\,dy$$

$$-\tau_{yx} \cdot dx \cdot dz + \left[\tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx}) \cdot dy\right] dx\,dz$$



$$\left[F_p + F_v\right]_x = \frac{\partial}{\partial x}(\sigma_{xx}) \cdot dx\,dy\,dz + \frac{\partial}{\partial z}(\tau_{zx}) \cdot dx\,dy\,dz + \frac{\partial}{\partial y}(\tau_{yx})\,dx\,dy\,dz$$

$$\left[V = \text{Vol of body} = dx \cdot dy \cdot dz\right]$$

$$\boxed{\left[F_p + F_v\right]_x = V \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}\tau_{zx}\right]}$$

In x direction:

$$F_{P_x} + F_{V_x} = \frac{\partial}{\partial x}(\sigma_{xx})\cdot V + \frac{\partial}{\partial y}(\tau_{yx})\cdot V + \frac{\partial}{\partial z}(\tau_{zx})\cdot V$$

y-direction,

$$F_{P_y} + F_{V_y} = \frac{\partial}{\partial x}(\tau_{xy})\cdot V + \frac{\partial}{\partial y}(\sigma_{yy})\cdot V + \frac{\partial}{\partial z}(\tau_{zy})\cdot V$$

z direction,

$$F_{P_z} + F_{V_z} = \frac{\partial}{\partial x}(\tau_{xz})\cdot V + \frac{\partial}{\partial y}(\tau_{yz})\cdot V + \frac{\partial}{\partial z}(\sigma_{zz})\cdot V$$

Stress matrix

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \Longrightarrow \text{Stress matrix}$$

* For irrotational flow, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

$$\boxed{F_{B_x} \to \dot{B}_x \qquad F_{B_y} \to \dot{B}_y \qquad F_{B_z} \to \dot{B}_z}$$

As per N.S eqⁿ, $\quad \dot{B}_x + (F_{p_x} + F_{v_x}) = m \cdot a_x$

x-direction

$$\Rightarrow \dot{B}_x + \frac{\partial}{\partial x}(\sigma_{xx}) \cdot V + \frac{\partial}{\partial y}(\tau_{yx}) \cdot V + \frac{\partial}{\partial z}(\tau_{zx}) \cdot V = \rho \cdot V \cdot \frac{Du}{Dt}$$

$$\Rightarrow \frac{\dot{B}_x}{V} + \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) = \rho \cdot \frac{Du}{Dt} \to A1$$

y

$$\frac{\dot{B}_y}{V} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\sigma_{yy}) + \frac{\partial}{\partial z}(\tau_{zy}) = \rho \cdot \frac{Dv}{Dt} \to A2$$

$$\frac{\dot{B}_z}{V} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\sigma_{zz}) = \rho \cdot \frac{Dw}{Dt} \to A3$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = f \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix}$$

[Stress & Strain]

Assumptions to N-S eq^n

(i) It obeys Newton's Law of viscosity, $\tau = \mu\left(\frac{\partial u}{\partial y}\right)$

(ii) It obeys Stoke's Law.

Solving the eq^n: Stress $\rightarrow f$(velocity component)

$$\sigma_{xx} = -P + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_{zz} = -P + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Substituting these stress values in eq^n (A1), (A2) & (A3)

$$\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right]^{*}$$

Continuous flow

$$\frac{\dot{B}_x}{V} - \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \cdot \frac{Du}{Dt}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

$$\left.\begin{aligned}
\frac{\dot{B}_x}{V} - \frac{\partial P}{\partial x} + \mu (\nabla^2 u) &= \rho \cdot \frac{Du}{Dt} \\
\frac{\dot{B}_y}{V} - \frac{\partial P}{\partial y} + \mu \cdot (\nabla^2 \cdot v) &= \rho \cdot \frac{Dv}{\partial t} \\
\frac{\dot{B}_z}{V} - \frac{\partial P}{\partial z} + \mu (\nabla^2 w) &= \rho \cdot \frac{Dw}{\partial t}
\end{aligned}\right\} \longrightarrow \text{N-S eq}^n$$