Module: 02
FLUID MECHANICS

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Outlines:

- Poiseuille's equation for flow of liquid through a tube.

15.14 POISEUILLE'S EQUATION FOR LIQUID-FLOW THROUGH A NARROW TUBE

In deducing an expression for the rate of flow of a liquid through a narrow tube, *Poiseuille* made the following assumptions:

(i) The liquid-flow is steady or streamline, with the streamlines parallel to the axis of the tube.

(ii) Since there is no radial flow, the pressure, in accordance with Bernoulli's theorem, is constant over any given cross-section of the tube.

(iii) The liquid in contact with the walls of the tube is stationary.

All these assumptions are found to be quite valid if the tube be narrow and velocity of liquid-flow really small.

Remembering further that a liquid yields to the smallest shearing stress and taking the tube to be horizontal to eliminate the effect of gravity on the liquid-flow, we may proceed to deduce Poiseuille's equation as follows:

Let a liquid of coefficient of viscosity η be flowing through a narrow horizontal tube of *radius* r and *length* l and when the conditions become steady, let the velocity of flow at all points on an imaginary, coaxial cylindrical shell of the liquid, of radius x, be v (Fig. 15.13) and, therefore, the velocity gradient, dv/dx.

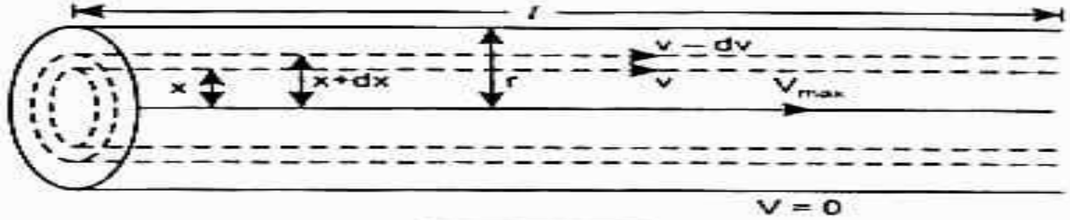

Fig. 15.13

Since the velocity of the liquid in contact with the walls of the tube is *zero* and goes on increasing as the axis is approached, where it is the maximum, it is clear that the liquid layer just inside the imaginary shell is moving faster, and the one just outside it, slower, than it. So that, in accordance with *Newton's law of viscous flow*, the *background dragging force on the imaginary liquid shell* is given by $F = \eta A dv/dx = 2\pi x l \eta dv/dx$, where A is the *surface area* of the shell, equal to $2\pi x l$.

And, if the pressure difference across the two ends of the tube be P, the *force on the liquid shell, accelerating it forwards* $= P \times \pi x^2$, where πx^2 is the area of cross-section of the shell.

For the liquid-flow to be steady, therefore, we must have driving force equal to backward dragging force, i.e., $P \times \pi x^2 = -2\pi x l \eta\ dv/dx$, the $-ve$ sign of the dragging force indicating that it acts in a direction opposite to that of the driving force. We, therefore, have

$$dv = -\frac{P\pi x^2 dx}{2\pi x l \eta} = -\frac{P x dx}{2\eta l}, \text{ which, on integration, gives}$$

$$v = -\frac{P}{2\eta l}\int x dx = -\frac{P}{2\eta l}\cdot\frac{x^2}{2} + C_1,$$

where C_1 is a constant of integration.

Since at $x = r$, $v = 0$, we have $0 = -\frac{Pr^2}{4\eta l} + C_1$. Or, $C_1 = \frac{Pr^2}{4\eta l}$.

$\therefore$ velocity of flow at distance x from the axis of the tube, *i.e.*,

$$v = -\frac{Px^2}{4\eta l} + \frac{Pr^2}{4\eta l} = \frac{P}{4\eta l}(r^2 - x^2).$$

which, incidentally shows at one that the profile or the velocity distribution curve of the advancing liquid is a parabola [as shown in Fig.15.1(b)], the velocity increasing from zero at the walls of the tube to a maximum at its axis.

Now, if we imagine another coaxial cylindrical shell of the liquid, of radius $x + dx$, enclosing the shell of radius x, the cross-sectional area between the two is clearly $2\pi x dx$ and, therefore, volume of the liquid flowing per second through this area is, say, $dQ = 2\pi x dx v$.

Imagining the whole of the liquid inside the tube to consists of such coaxial cylindrical shells, the volume of the liquid flowing through all of them per second *i.e.*, the *rate of flow through the tube, as a whole* say, Q, is obtained by integrating the expression for dQ between the limits $x = 0$ and $x = r$. We thus have rate of liquid flow through the narrow tube, *i.e.*,

$$Q = \int_0^r 2\pi x dx v = \int_0^r 2\pi x \frac{P}{4\eta l}(r^2 - x^2)dx$$

$$= \frac{\pi P}{2\eta l}\left[\frac{x^2 r^2}{2} - \frac{x^4}{4}\right]_0^r = \frac{\pi P}{2\eta l}\left(\frac{r^4}{2} - \frac{r^4}{4}\right) = \frac{\pi P}{2\eta l}\cdot\frac{r^4}{4}.$$

Or, $$Q = \frac{\pi P r^4}{8\eta l} \quad \text{...(I)}$$

whence, $$\eta = \frac{\pi P r^4}{8Ql} \quad \text{...(II)}$$

Thank You