MOMENTUM-INTEGRAL EQUATIONS FOR BOUNDARY LAYER

- Karman and Pohlhausen devised a simplified method by satisfying only the boundary conditions of the boundary layer flow rather than satisfying Prandtl's differential equations for each and every particle within the boundary layer.

* Consider, steady, two dimensional and incompressible flow.

i.e

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy = \left(-\frac{1}{\rho}\frac{dP}{dx} + v\frac{\partial^2 u}{\partial y^2}\right)dy$$

[from eqⁿ ⑥ & ⑦ of boundary layer Equation]

Now,

Integrating with respect to $y=0$ (wall) & $y=\delta$,

$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy = \int_0^\delta \left(-\frac{1}{\rho}\frac{dP}{\partial x} + v\frac{\partial^2 u}{\partial y^2}\right)dy$$

$$\Rightarrow \int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta -\frac{1}{\rho}\frac{\partial P}{\partial x}dy + \int_0^\delta v\frac{\partial^2 u}{\partial y^2}dy \rightarrow (1)$$

Now,

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = [vu]_0^\delta - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$\Rightarrow \int_0^\delta v\frac{\partial u}{\partial y}dy = U_\infty v_\delta + \int_0^\delta u\frac{\partial u}{\partial x}dy$$

$\left[\because \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}\right.$ by continuity eqⁿ$\left.\right]$

$$\Rightarrow \int_0^\delta v\frac{\partial u}{\partial y}dy = -U_\infty \int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy \rightarrow (2)$$

Substituting these value in equation (1)

$$\int_0^{\delta} u\frac{\partial u}{\partial x}dy - U_\infty\int_0^{\delta}\frac{\partial u}{\partial x}dy + \int_0^{\delta} u\frac{\partial u}{\partial x}dy = -\int_0^{\delta}\frac{1}{\rho}\frac{\partial p}{\partial x}dy + \int_0^{\delta}\nu\frac{\partial^2 u}{\partial y^2}dy$$

$$\Rightarrow \int_0^{\delta} 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^{\delta}\frac{\partial u}{\partial x}dy = -\int_0^{\delta}\frac{1}{\rho}\frac{dp}{dx}dy - \nu\frac{\partial u}{\partial y}\bigg|_{y=0} \rightarrow (3)$$

Substituting the ~~equation~~ relation between $\frac{\partial p}{\partial x}$ and free stream velocity U_∞ for the inviscid zone in eq (3),

$$\int_0^{\delta} 2u\frac{\partial u}{\partial x}dy - U_\infty\int_0^{\delta}\frac{\partial u}{\partial x}dy - \int U_\infty\frac{dU_\infty}{dx}dy = -\left(\frac{\mu\frac{\partial u}{\partial y}\big|_{y=0}}{\rho}\right)$$

$$\Rightarrow \int_0^{\delta}\left(2u\frac{\partial u}{\partial x} - U_\infty\frac{\partial u}{\partial x} - U_\infty\frac{dU_\infty}{dx}\right)dy = -\frac{\tau_w}{\rho}$$

$$\cancel{\Rightarrow \int_0^{\delta}\left(-u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial x} + U_\infty\frac{\partial u}{\partial x} \pm U_\infty\frac{dU_\infty}{dx}\right)dy = \frac{\tau_w}{\rho}}$$

$$\Rightarrow \int_0^{\delta}\frac{\partial}{\partial x}[u(U_\infty - u)]dy + \frac{dU_\infty}{dx}\int_0^{\delta}(U_\infty - u)dy = \frac{\tau_w}{\rho}$$

Since the integrals vanish outside the boundary layer, so, increase the integration limit i.e $\delta = \infty$

$$\int_0^{\infty}\frac{\partial}{\partial x}[u(U_\infty - u)]dy + \frac{dU_\infty}{dx}\int_0^{\infty}(U_\infty - u)dy = \frac{\tau_w}{\rho}$$

$$\Rightarrow \frac{d}{dx}\int_0^{\infty}[u(U_\infty - u)]dy + \frac{dU_\infty}{dx}\int_0^{\infty}(U_\infty - u)dy = \frac{\tau_w}{\rho} \rightarrow (4)$$

Substituting displacement thickness and momentum thickness in eq (4)

$$\frac{d}{dx}[U_\infty^2\delta^{**}] + \delta^{*}U_\infty\frac{dU_\infty}{dx} = \frac{\tau_w}{\rho} \rightarrow (5)$$

Equation (5) is known as the momentum integral equation for two dimensional incompressible laminar boundary layer.