- ▢ Principles of Telecommunication
- ▢ Module 5
- ▢ Sampling Theory and Analog Pulse Modulation techniques PAM, PPM, PWM
- ▢ Lecture Plan- PART 1

1. Sampling theory
2. Nyquist rate and Nyquist interval
3. Analog Pulse Modulation methods - Classification of PAM, PPM, PWM

1. Sampling Theorem

Definition

"A continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is fs≥ 2fm. Here fs is the sampling frequency and fm is the maximum frequency present in the signal".

The statement of sampling theorem can be given in two parts as:
(i) A band-limited signal of finite energy, which has no frequency-component higher than fm Hz, is completely described by its sample values at uniform intervals less than or equal to 1/ 2fm second apart.
(ii) A band-limited signal of finite energy, which has no

frequency components higher than fm Hz, may be completely recovered from the knowledge of its samples taken at the rate of 2fm samples per second.

- The first part represents the representation of the signal in its samples and minimum sampling rate required to represent a continuous-time signal into its samples.

- The second part of the theorem represents reconstruction of the original signal from its samples. It gives sampling rate required for satisfactory reconstruction of signal from its samples.

Proof of Sampling Theorem

To prove the sampling theorem, we need to show that a signal whose spectrum is band-limited to fm Hz, can be reconstructed exactly without any error from its samples taken uniformly at a rate fs > 2 fm Hz.

Let us consider a continuous time signal x(t) whose spectrum is band-limited to fm Hz. This means that the signal x(t) has no frequency components beyond fm Hz.

$$X(\omega) = 0 \text{ for } |\omega| > \omega_m$$

Therefore, $X(\omega)$ is zero for $|\omega| > \omega_m$, i.e., where $\omega_m = 2\pi f_m$

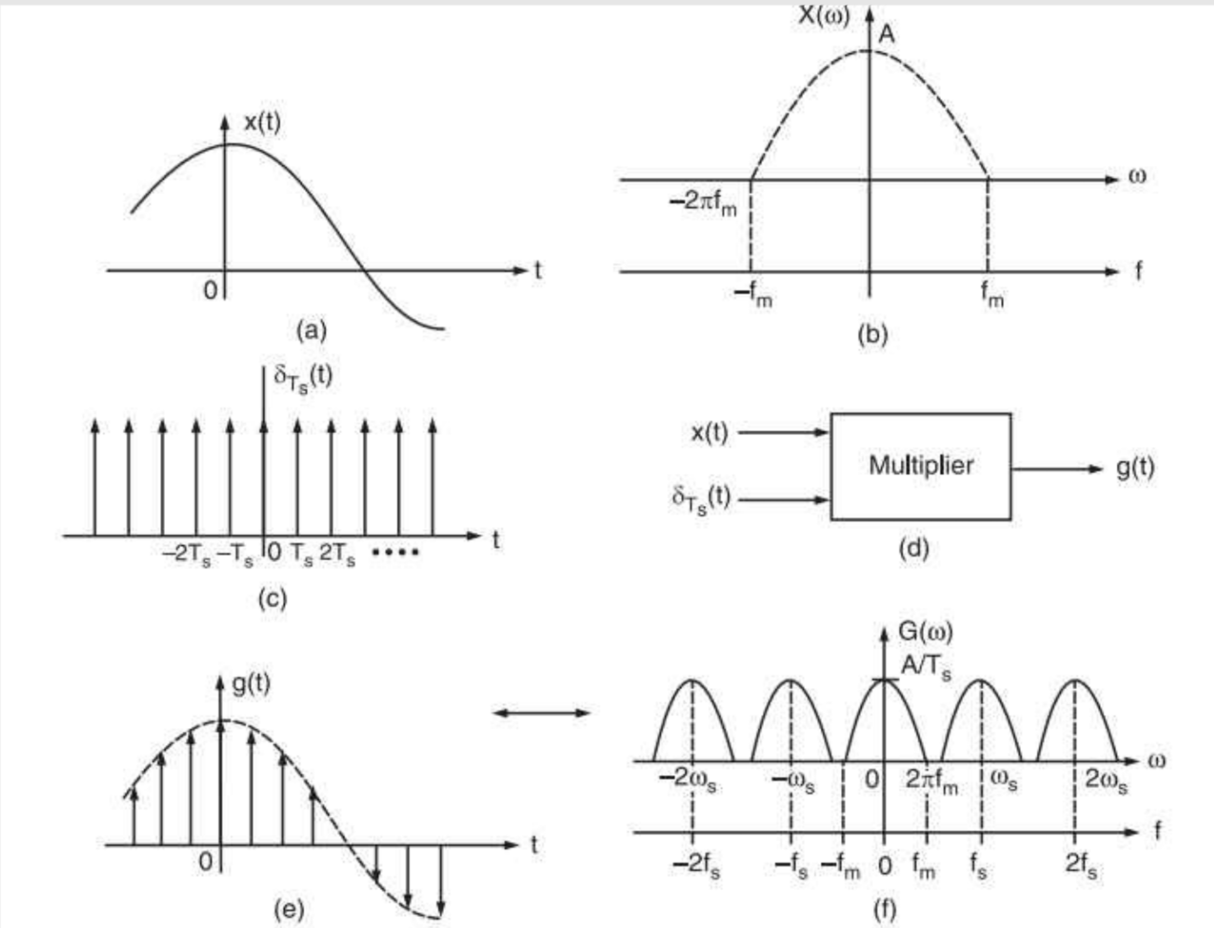


Fig.1 (a) shows this continuous-time signal x(t).

Fig.1 (a) A continuous-time signal, (b) Spectrum of continuous-time signal, (c) Impulse train as sampling function, (d) Multiplier, (e) Sampled signal, (f) Spectrum of sampled signal.

- [ ] Let X(ω) represents its Fourier transform or frequency spectrum as shown in fig.1(b).
- [ ]

Sampling of x(t) at a rate of fs Hz (i.e.,fs samples per

second) can be achieved by multiplying x(t) by an impulse train δTs(t). The impulse train δTs(t) consists of unit impulses repeating periodically
every Ts seconds, where Ts = 1/fs.

- Fig.1(c) shows this impulse train. This multiplication results in the sampled signal g(t) shown in fig.1(d).

This sampled signal consists of impulses spaced every Tsseconds (the sampling interval). The resulting or sampled

$$g(t) = x(t)\, \delta_{T_s}(t)$$

signal may be written as under.
................(1)
Again, since the impulse train δTs(t) is a periodic signal of period Ts, it may be expressed as a Fourier series. The Fourier series expansion of impulse-train δTs(t) may be

$$\delta_{T_s}(t) = \frac{1}{T_s}\,[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \ldots]$$

expressed as under:

................(2)

$$\omega_s = \frac{2\pi}{T_s} = 2\,\pi f_s$$

Here,

Putting the values of δTs(t) from equation (2) in equation (1),

$$g(t) = \frac{1}{T_s} [x(t) + 2\,x(t) \cos \omega_s t + 2x(t) \cos 2\,\omega_s t + 2x(t) \cos 3\,\omega_s t + ...]$$

the sampled signal can be written as,

................(3)

Now, to obtain G(ω), the Fourier transformation of g(t), we will have to take the Fourier transform of right hand side. Thus, Fourier transform of x(t) is X(w).

Fourier transform of 2x(t) cos ωst is [X(ω – ωs) + X(ω + ωs)]. Fourier transform of 2x(t) cos 2ωst is [X(ω – 2ωs) + X (ω + 2ωs)] and so on. Therefore, on taking Fourier transformation, the equation (3) becomes,

$$G(\omega) = \frac{1}{T_s} [X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\,\omega_s) + X(\omega + 2\,\omega_s) + X(\omega - 3\,\omega_s) + X(\omega + 3\,\omega_s) + ...] \quad ...$$

................(4)

Or

$$G(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\,\omega_s)$$

................(5)

From equations (4) and (5), it is quite obvious that the spectrum G(ω) consists of X(ω) repeating periodically with period $\omega s = 2\pi / Ts$ rad/sec. or $fs = 1 / Ts$ Hz as shown in fig.1 (f).

Now, if have to reconstruct x(t) from g(t), we must be able to recover X(ω) from G(ω). This is possible if there is no overlap between successive cycles of G(ω). Fig.1 (f) shows

$$f_s > 2 f_m$$

that this requires,

But, the sampling interval $Ts = 1 / fs$

$$T_s < \frac{1}{2f_m}$$

Hence,

Therefore, as long as the sampling frequency fs is greater than twice the maximum signal frequency fm (signal bandwidth, fm), G(ω) will consist of non-overlapping repetitions of X(ω). If this is true, fig.1 (f) shows that x(t) can be recovered from its samples g(t) by passing the sampled signal g(t) through an ideal low-pass filter (LPF) of bandwidth fm Hz. This proves the sampling theorem.

2. Nyquist Rate and Nyquist interval

When the sampling rate becomes exactly equal to 2fm samples per second, then it is called Nyquist rate. Nyquist rate is also called the minimum sampling rate. It is

$$f_s = 2\,f_m$$

given

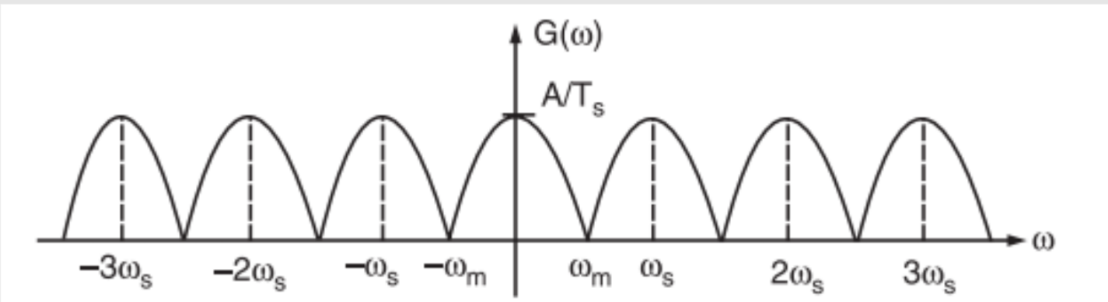
Similarly, maximum sampling interval is called Nyquist

$$T_s = \frac{1}{2f_m}\ \text{seconds}$$

interval. It is given by,

When the continuous-time band-limited signal is sampled at Nyquist rate (fs = 2fm), the sampled-spectrum G(ω) contains non-overlapping G(ω) repeating periodically.

But the successive cycles of G(ω) touch each other as shown in fig.1. Therefore, the original spectrum X(ω) can be



recovered from the sampled spectrum by using a low pass filter (LPF) with a cut-off frequency ωm.

Fig.1: Sampled spectrum at Nyquist rate

EXAMPLE 9.1. An analog signal is expressed by the equation $x(t) = 3 \cos (50 \pi t) + 10 \sin (300\pi t) - \cos (100\pi t)$. Calculate the Nyquist rate for this signal.

Solution : The given signal is expressed as

$$x(t) = 3 \cos 50\pi t + 10 \sin 300\pi t - \cos 100\pi t \quad ...(i)$$

Let three frequencies present be ω_1, ω_2 and ω_3

So that the new equation for signal,

$$x(t) = 3 \cos \omega_1 t + 10 \sin \omega_2 t - \cos \omega_3 t \quad ...(ii)$$

Comparing equations (i) and (ii) we have

$$\omega_1 t = 50 \pi t; \omega_1 = 50 \pi$$

or $$2 \pi f_1 = 50 \pi$$

or $$2 f_1 = 50$$

Hence, $$f_1 = 25 \text{ Hz}$$

Similarly, for second factor $$\omega_2 t = 300 \pi t \quad \text{or} \quad \omega_2 = 300\pi$$

or $$2 \pi f_2 = 300 \pi$$

or $$2 \pi f_2 = 300 \pi$$

Therefore, $$f_2 = 150 \text{ Hz.}$$

Again, for third factor $$\omega_3 t = 100 \pi t \text{ or } 2 \pi f_3 t = 100 \pi t$$

or $$2 \pi f_3 = 100 \pi$$

Therefore, $$f_3 = 50 \text{ Hz}$$

Therefore, the maximum frequency present in x(t) is,

$$f_2 = 150 \text{ Hz}$$

Nyquist rate is given as $$f_s = 2 f_m$$

where f_m = Maximum frequency present in the signal.

$$\text{Here } f_m = f_2 = 150 \text{ Hz}$$

Therefore, Nyquist rate $$f_s = 2 f_2 = 2 \times 150 = 300 \text{ Hz}$$ **Ans.**

EXAMPLE 9.2. Find the Nyquist rate and the Nyquist interval for the signal

$$x(t) = \frac{1}{2\pi} \cos (4000\pi t) \cos (1000\pi t)$$

Solution : Given signal is

$$x(t) = \frac{1}{2\pi} \cos (4000\pi t) \cos (1000\pi t)$$

or $$x(t) = \frac{1}{4\pi} [2 \cos (4000\pi t) \cos 1000\pi t)]$$

or $$x(t) = \frac{1}{4\pi} [\cos (4000\pi t + 1000\pi t) + \cos (4000\pi t - 1000\pi t)]$$

[Since, $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$]

or $$x(t) = \frac{1}{4\pi} [\cos 5000\pi t + \cos 3000\pi t] \quad ...(i)$$

Ref- Problems: Book Dr. Sanjay Sharma

Example 9.2: Common question- 2019 previous years paper

Let the two frequencies present in the signal be ω_1 and ω_2 so that the new equation for the signal will be

$$x(t) = \frac{1}{4\pi}[\cos \omega_1 t + \cos \omega_2 t] \quad ...(ii)$$

Comparing equations (i) and (ii), we have

$$\omega_1 t = 5000\pi t$$

or $$2\pi f_1 t = 5000\pi t$$

or $$2 f_1 = 5000$$

Therefore, $$f_1 = 2500 \text{ Hz}$$

Similarly, for second factor

$$\omega_2 t = 3000\pi t$$

or $$2\pi f_2 t = 3000\pi t$$

or $$2\pi f_2 = 3000$$

Hence, $$f_2 = 1500 \text{ Hz}$$

Therefore, the maximum frequency present in x(t) is

$$f_1 = 2500 \text{ Hz}$$

Nyquist rate is given as

$$f_s = 2 f_m$$

where f_m = Maximum frequency present in the signal.

Here, $$f_m = f_1 = 2500 \text{ Hz}$$

Therefore Nyquist rate

$$f_s = 2 f_m = 2 \times 2500 = 5000 \text{ Hz} = 5 \text{ kHz}$$ **Ans.**

Nyquist interval is given as

$$T_s = \frac{1}{2f_m} = \frac{1}{2 \times 2500} = \frac{1}{5000}$$

or $$T_s = 0.2 \times 10^{-3} \text{ seconds} = 0.2 \text{ m sec.}$$ **Ans.**

3. Analog pulse modulation methods

- <u>In previous topics</u>, in analog modulation systems, some parameter of a sinusoidal carrier is varied according to the instantaneous value of the modulating signal.

- While, in analog pulse modulation methods, thee carrier is no longer continuous signal but consists of a pulse train.

- Some parameter of which is varied according to the instantaneous value of the modulating signal.

- There are two types of pulse Modulation systems as under:

1. Pulse Amplitude Modulation (PAM)
2. Pulse Time Modulation (PTM)

In pulse amplitude modulation (PAM), the amplitude of the carrier is varied in accordance with the modulating signal.
Whereas, in PTM, the timing of the carrier pulse train is varied .

There are two types of PTM
(a) Pulse width modulation (PWM)
(b) Pulse position modulation (PPM)