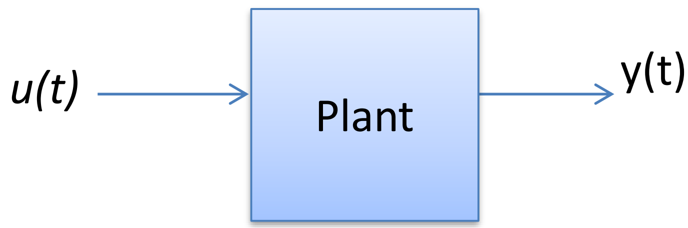
MODULE 2:
Models of Physical Systems

Transfer function: definition and properties, poles, zeros and pole-zero map, formulation of differential equations for physical systems and derivation of transfer function: mechanical and electrical systems, derivation of transfer function using block diagrams reduction techniques and signal flow graphs, signal flow graph from block diagram, analogous systems.

Transfer Function

- Transfer Function is the ratio of Laplace transform of the output to the Laplace transform of the input. Consider all initial conditions to zero.



$$If \qquad L\{u(t)\} = U(S) \qquad and$$

$$L\{y(t)\} = Y(S)$$

- Where L is the Laplace operator.

Transfer Function

- The transfer function G(S) of the plant is given as

$$G(S) = \frac{Y(S)}{U(S)}$$

- Why input, output and other signals are represented in Laplace form in a control system?

The input and output of a control system can be different types.

For mathematical analysis of a system, all kinds of signal should be represented in similar form.

- **Laplace Transformation:**

  Laplace transformation is a technique for solving differential equations.

  Differential equation of time domain form is first transformed to algebraic equation of frequency domain form.

  After solving the algebraic equation in frequency domain, the result then is finally transformed to time domain form to achieve the ultimate solution of the differential equation.

  In other words it can be said that the Laplace transformation is nothing but a shortcut method of solving differential equation.

TRANSFER FUNCTION

Definition: The transfer function is defined as the ratio of Laplace transform of the output to the Laplace transform of input with all initial conditions are zero.

We can defined the transfer function as

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots.. + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots.. + a_1 s + a_0} ---------- (1.0)$$

- In equation (1.0), if the order of the denominator polynomial is greater than the order of the numerator polynomial then the transfer function is said to be STRICTLY PROPER.
- If the order of both polynomials are same, then the transfer function is PROPER.
- The transfer function is said to be IMPROPER, if the order of numerator polynomial is greater than the order of the denominator polynomial.

CHARACTERISTIC EQUATION:

The characteristic equation can be obtained by equating the denominator polynomial of the transfer function to zero. That is

$$a_n s^n + a_{n-1} s^{n-1} + \ldots.. + a_1 s + a_0 = 0$$

POLES AND ZEROS OF A TRANSFER FUNCTION

POLES : The poles of G(s) are those values of ‘s’ which make G(s) tend to infinity.

ZEROS: The zeros of G(s) are those values of ‘s’ which make G(s) tend to zero.

If either poles or zeros coincide, then such type of poles or zeros are called multiple poles or zeros, otherwise they are known as simple poles or zeros.

- For example, consider following transfer function

$$G(s) = \frac{50(s+3)}{s(s+2)(s+4)^2}$$

This transfer function having the simple poles at s=0, s=-2, multiple poles at s=-4 and simple zero at s=-3.

Basic Elements of Electrical Systems

Symbol →

- The time domain expression relating voltage and current for the resistor is given by Ohm's law i-e

$$v_R(t) = i_R(t)R$$

- The Laplace transform of the above equation is

$$V_R(s) = I_R(s)R$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the Capacitor is given as:

$$v_c(t) = \frac{1}{C}\int i_c(t)dt$$

- The Laplace transform of the above equation (assuming there is no charge stored in the capacitor) is

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Basic Elements of Electrical Systems

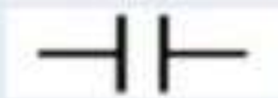
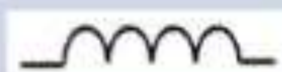
- The time domain expression relating voltage and current for the inductor is given as:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

- The Laplace transform of the above equation (assuming there is no energy stored in inductor) is

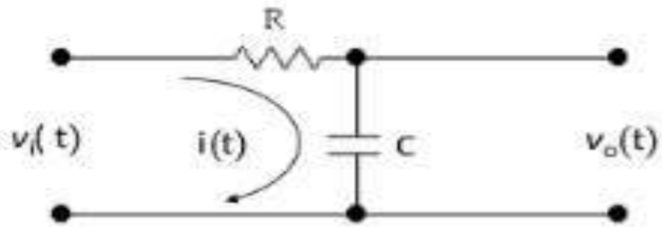
$$V_L(s) = LsI_L(s)$$

V-I and I-V relations

Component	Symbol	V-I Relation	I-V Relation
Resistor		$v_R(t) = i_R(t)R$	$i_R(t) = \frac{v_R(t)}{R}$
Capacitor		$v_c(t) = \frac{1}{C}\int i_c(t)dt$	$i_c(t) = C\frac{dv_c(t)}{dt}$
Inductor		$v_L(t) = L\frac{di_L(t)}{dt}$	$i_L(t) = \frac{1}{L}\int v_L(t)dt$

Example#1

- The two-port network shown in the following figure has $v_i(t)$ as the input voltage and $v_o(t)$ as the output voltage. Find the transfer function $V_o(s)/V_i(s)$ of the network.



$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

Example#1

$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt \qquad v_o(t) = \frac{1}{C}\int i(t)dt$$

- Taking Laplace transform of both equations, considering initial conditions to zero.

$$V_i(s) = I(s)R + \frac{1}{Cs}I(s) \qquad V_o(s) = \frac{1}{Cs}I(s)$$

- Re-arrange both equations as:

$$V_i(s) = I(s)(R + \frac{1}{Cs}) \qquad CsV_o(s) = I(s)$$

Example#1

$$V_i(s) = I(s)(R + \frac{1}{Cs}) \qquad CsV_o(s) = I(s)$$

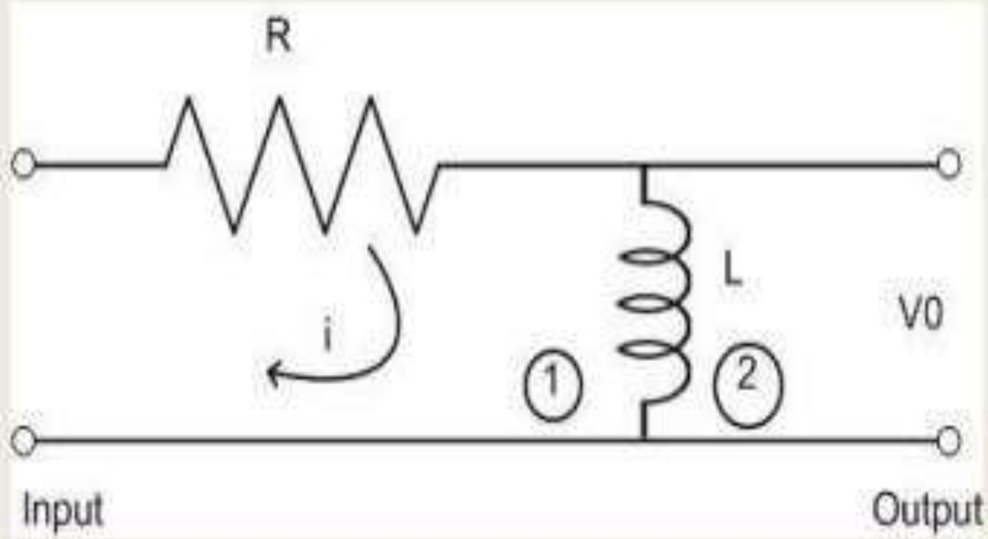
- Substitute *I(s)* in equation on left

$$V_i(s) = CsV_o(s)(R + \frac{1}{Cs})$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{Cs(R + \frac{1}{Cs})}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

Find the transfer function of the given figure.



Solution:

Step 1: Apply KVL in mesh 1 and mesh 2

$$v_i = Ri + L\frac{di}{dt} ----(1)$$

$$v_o = L\frac{di}{dt} ----(2)$$

Step 2: take Laplace transform of eq. (1) and (2)

$$V_i(s) = RI(s) + sLI(s) ------(3)$$

$$V_o(s) = sLI(s) ------(4)$$

Step 3: calculation of transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{sLI(s)}{(R+sL)I(s)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{sL}{R+sL} ------(5)$$

Equation (5) is the required transfer function

A system having input x(t) and output y(t) is represented by Equation (1). Find the transfer function of the system.

$$\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 5x(t) ------(1)$$

Solution: taking Laplace transform of equation (1)

$$sY(s) + 4Y(s) = sX(s) + 5X(s)$$

$$Y(s)(s+4) = X(s)(s+5)$$

$$\frac{Y(s)}{X(s)} = \frac{s+5}{s+4}$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s+5}{s+4}$$

G(s) is the required transfer function

The transfer function of the given system is given by

$$G(s) = \frac{4s+1}{s^2+2s+3}$$

Find the differential equation of the system having input x(t) and output y(t).

Solution:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{4s+1}{s^2+2s+3}$$

$$X(s)[4s+1] = Y(s)\left[s^2+2s+3\right]$$

$$4sX(s) + X(s) = s^2 + 2sY(s) + 3Y(s)$$

Taking inverse Laplace transform, we have

$$4\frac{dx(t)}{dt} + x(t) = \frac{dy^2}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t)$$

Required differential equation is

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = 4\frac{dx(t)}{dt} + x(t)$$

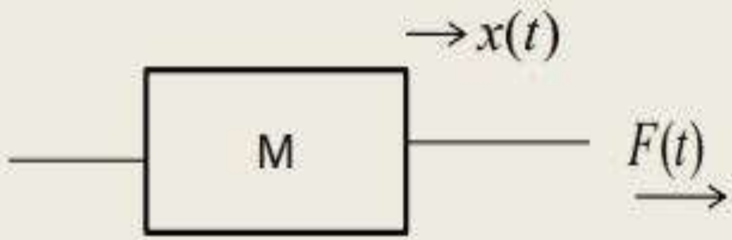
MECHANICAL SYSTEM

- **TRANSLATIONAL SYSTEM:** The motion takes place along a strong line is known as translational motion. There are three types of forces that resists motion.
- **INERTIA FORCE:** consider a body of mass ‘M’ and acceleration ‘a’, then according to Newton’s law of motion

$$F_M(t)=Ma(t)$$

If v(t) is the velocity and x(t) is the displacement then

$$F_M(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2}$$



- **DAMPING FORCE:** For viscous friction we assume that the damping force is proportional to the velocity.

$$F_D(t) = B\,v(t) = B\frac{dx(t)}{dt}$$

Where B= Damping Coefficient in N/m/sec.

We can represent ‘B’ by a dashpot consists of piston and cylinder.

- **SPRING FORCE:** A spring stores the potential energy. The restoring force of a spring is proportional to the displacement.

$$F_K(t) = \alpha x(t) = K\,x(t)$$

$$F_K(t) = K\int v(t)\,dt$$

Where 'K' is the spring constant or stiffness (N/m)

The stiffness of the spring can be defined as restoring force per unit displacement

- **ROTATIONAL SYSTEM:** The rotational motion of a body can be defined as the motion of a body about a fixed axis. There are three types of torques resists the rotational motion.
- **Inertia Torque:** Inertia(J) is the property of an element that stores the kinetic energy of rotational motion. The inertia torque is the product of moment of inertia J and angular acceleration α(t).

$$T_I(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

Where ω(t) is the angular velocity and θ(t) is the angular displacement.

- **Damping torque:** The damping torque TD(t) is the product of damping coefficient B and angular velocity ω. Mathematically

$$T_D(t) = B\omega(t) = B\frac{d\theta(t)}{dt}$$

- Spring torque: Spring torque Tθ(t) is the product of stiffness and angular displacement. Unit of 'K' is N-m/rad

$$T_\theta(t) = K\theta(t)$$