- ▢ Principles of Telecommunication
- ▢ Module 4
- ▢ Probability, Random Signals and Random Process_ PDF, CDF
- ▢ Lecture Plan- Part2

1. Probability Density Function (PDF)

The cumulative distribution function (CDF) can give useful information about discrete as well as continuous random variables. However, the probability density function (PDF) is a more convenient way of describing a continuous random variable. The probability density function fX(x) is defined as the derivative of the cumulative distribution function. Thus, we have,

$$\text{PDF: } f_X(x) = \frac{d}{dx} F_X(x)$$

Properties of PDF

Property 1: The CDF can be derived from PDF by integrating it i.e.,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

Proof :

According to the definition of PDF , we have

$$\text{PDF: } f_X(x) = \frac{d}{dx} F_X(x)$$

And Integrating on both the sides.
It is important to note the upper limit of integration. It is not + ∞ but, it is 'x'. This is because FX(x) has been defined as the probability of X ≤ x.

Thus,

$$\int_{-\infty}^{x} f_X(x)\,dx = \left[F_X(x)\right]_{-\infty}^{x} = [F_X(x) - F_X(-\infty)]$$

But, FX(− ∞) = 0
Therefore,

$$\int_{-\infty}^{x} f_X(x)\,dx = F_X(x) = 0$$

or,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

Property 2 : PDF is a non-negative function for all values of x i.e.,

fX(x) ≥ 0   for all x

Reasoning: As we know that CDF is a monotone increasing function. PDF is the derivative of CDF and the derivative of a monotone increasing function will always be positive.

Property 3: The area under PDF curve is always equal to unity.

Therefore,

$$\int_{-\infty}^{x} f_X(x)\,dx = 1$$

Proof :

As per the definition of PDF, we have

$$f_X(x)\,dx = \frac{d}{dx} F_X(x)$$

Integrating both sides, we get

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)\,dx = \left[F_X(x)\right]_{-\infty}^{\infty} = F_X(\infty) - F_X(-\infty)$$

But,

$$F_X(-\infty) = 0 \quad \text{and} \quad F_X(\infty) = 1$$

Therefore,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 - 0 = 1$$

2. Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of a random variable is defined as the probability that the random variable X takes value less than or equal to x.
ie., CDF $F_X(x) = P(X \leq x)$ ………………(1)

Here, x is a dummy variable and the CDF is denoted by $F_X(x)$.

It is possible to define CDF for continuous as well as the discrete random variables. The CDF is sometimes called as simply the distribution function.

Important Properties of CDF

Property 1: The CDF is always bounded between 0 and 1.

i.e., $0 \leq F_X(x) \geq 1$ …………..(2)

As per the definition of CDF, it is a probability function $P(X \leq x)$ and any probability must have a value between 0 and 1. Therefore, CDF is always bounded between 0 and 1.

Property 2: This property states that,

$F_X(\infty)= 1$ …………(3)

Proof: Here, $F_X(\infty) = P(X \leq \infty)$. This includes the probability of all the possible outcomes or events. The random variable $X \leq \infty$, thus, becomes a 'certain event' and therefore has a 100% probability.

Property 3: This property states that,

$F_X(-\infty) = 0$ ……………..(6.18)

Proof: Here, $F_X(-\infty) = P(X \leq -\infty)$. The random variable X cannot have any value which is less than or equal to $-\infty$. Thus, $X \leq -\infty$ is a null event and therefore, has a 0% probability.

Property 4: This property states that $F_X(x)$ is a monotone non-decreasing function i.e.,

$F_X(x_1) \leq F_X(x_2)$ for $x_1 < x_2$ ………………(4)

Proof: To prove this property, let us consider fig.1.

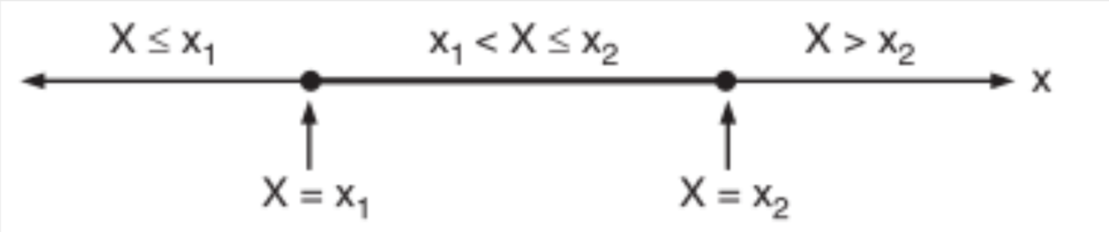

Fig.1

FX(x2) is defined as under: FX(x2) = P(X ≤ x2)

The R.H.S of this equation can be expressed as union of two probabilities.

Therefore,

FX(x2) = P(X ≤ x2) = P(X ≤ x1) ∪ P (x1 < X ≤ x2) ………………. (5)

The two events X ≤ x1 and x1 < X ≤ x2 are mutually exclusive.

Therefore, we write

P(X ≤ x1) ∪ P (x1 < X ≤ x2) = P(X ≤ x1) + P (x1 < X ≤ x2)

Substituting this value in equation (5), we get

FX(x2) = P(X ≤ x1) + P(x1 < X ≤ x2) …………….(6)

But, the first term in equation (6) i.e., $P(X \leq x_1) = F_X(x_1)$

Therefore,

$F_X(x_2) = F_X(x_1) + P(x_1 < X \leq x_2)$ ………………(7)

$P(x_1 < X \leq x_2)$ is always non-negative as it is a probability function, Thus,

$$F_X(x_2) \geq F_X(x_1)$$

or $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$

These properties of CDF are general and are valid for the continuous as well as discrete random variables.