We have to learn,

- Del Operator
- Gradient of scalar function
- Directional Derivative
- Divergence of a vector
- Curl of a vector

Del Operator

- Operator ∇ is called vector differential operator defined as

$$\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$$

Gradient of scalar function

- If $\phi(x, y, z)$ is a scalar function of three variables and ϕ is differentiable, the gradient of ϕ is defined as

$$grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Where ,

ϕ is a scalar function

$\nabla\phi$ is a vector function

If $\phi = x^2yz^3 + xy^2z^2$,determine $grad\phi$ at point P=(1,3,2).

<u>solution</u>

$$\phi = x^2yz^3 + xy^2z^2$$

$$\frac{\partial \phi}{\partial x} = 2xyz^3 + y^2z^2, \frac{\partial \phi}{\partial y} = x^2z^3 + 2xyz^2, \frac{\partial \phi}{\partial z} = 3x^2yz^2 + 2xy^2z$$

Therefore,

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= \left(2xyz^3 + y^2z^2\right)\hat{i} + \left(x^2z^3 + 2xyz^2\right)\hat{j} + \left(3x^2yz^2 + 2xy^2z\right)\hat{k}$$

at

$$P = (1,3,2)$$

$$\nabla \phi = 84\hat{i} + 32\hat{j} + 72\hat{k}$$

Grad Properties

If A and B are two scalars ,then

1) $\nabla(A \pm B) = \nabla A \pm \nabla B$

2) $\nabla(AB) = A(\nabla B) + B(\nabla A)$

Directional Derivative

Directional derivative of ϕ in the direction of a is

$$\frac{d\phi}{ds} = \hat{a}.grad\phi$$

where,

$$\hat{a} = \frac{dr}{|dr|}$$

Which is a unit vector in the direction of dr .

Compute the directional derivative of $\phi = x^2z + 2xy^2 + yz^2$ at the point (1,2,-1) in the direction of the vector A=2i+3j-4k.

Solution

Directional derivative of ϕ in the direction of a

$$\frac{d\phi}{ds} = \hat{a}.grad\,\phi$$

Where,

$$grad\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

And , $\hat{a} = \frac{A}{|A|}$

$\phi = x^2z + 2xy^2 + yz^2$ Hence ,

$$\nabla\phi = \left(2xz + 2y^2\right)\hat{i} + \left(4xy + z^2\right)\hat{j} + \left(x^2 + 2yz\right)\hat{k}$$

$At(1,2,-1),$

$$\nabla\phi = 6\hat{i} + 9\hat{j} - 3\hat{k}$$

Also given $A = 2\hat{i} + 3\hat{j} - 4\hat{k}$,then

$$|A| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$$

$$Therefore,$$

$$\hat{a} = \frac{A}{|A|} = \frac{1}{\sqrt{29}}\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right)$$

$$then,$$

$$\frac{d\phi}{ds} = \hat{a}.grad\,\phi$$

$$= \frac{51}{\sqrt{29}}$$

Divergence of a vector

If $A = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$,the divergence of A is defined as

$$divA = \nabla . A$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) . \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \right)$$

$$\Rightarrow \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} .$$

If $A = x^2 y\hat{i} - xyz\hat{j} + yz^2\hat{k}$, determine $divA$ at point (1,2,3) .

solution

$$divA = \nabla . A$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) . \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \right)$$

$$\Rightarrow \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} .$$

$$divA = 2xy - xz + 2yz$$

$$at(1, 2, 3)$$

$$divA = 2(1)(2) - (1)(3) + 2(2)(3)$$

$$divA = 13$$

Curl of a vector

If $A = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$,the curl of A is defined by

$$curl A = \nabla \times A$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \right)$$

$$\Rightarrow curl A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

If $A=(y^4-x^2z^2)\hat{i}+(x^2+y^2)\hat{j}-x^2yz\hat{k}$,determine curl A at (1,3,-2).

<u>solution</u>

$$curlA=\nabla\times A=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^4-x^2z^2 & x^2+y^2 & -x^2yz\end{vmatrix}$$

$$=-x^2zi-(-2xyz+2x^2z)\hat{j}+(2x-4y^3)\hat{k}$$

$$At(1,3,-2),$$

$$curlA=2\hat{i}-8\hat{j}-106\hat{k}$$