Gamma Distribution:

The failure density function for gamma distribution is

$$f(t) = \frac{\lambda^\eta}{\Gamma(n)} t^{\eta-1} e^{-\lambda t} \quad t \geq 0, \eta > 0, \lambda > 0$$

where η is a shape parameter and λ is the scale parameter

$$F(t) = \int_0^t \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} d\tau$$

If η is an integer, it can be shown by successive integration by parts that

$$F(t) = \sum_{k=n}^{\infty} \frac{\left((\lambda t)^k \exp[-\lambda t]\right)}{k!}$$

Reliability function

$$R(t) = \sum_{k=0}^{\eta-1} \frac{\left((\lambda t)^k \exp[-\lambda t]\right)}{k!}$$

- An amplifier has an exponential time-to-failure distribution with a failure rate of 8% per 1000 hours. What is the reliability of the amplifier at 5000hours? Find the mean time to failure.

Solution The constant failure rate λ is obtained as

$$\lambda = 0.08/1000 \text{ hours} = 0.00008/\text{hour}$$

The reliability at 5000 hours is

$$R(t) = e^{-\lambda t} = e^{-(0.00008)(5000)} = e^{-0.4} = 0.6703$$

The mean time to failure is

$$\text{MTTF} = 1/\lambda = 1/0.00008 = 12,500 \text{ hours}$$

- What is the highest failure rate for a product if it is to have a probability of survival (i.e., successful operation) of 95% at 4000 hours? Assume that the time to failure follows an exponential distribution.

Solution The reliability at 4000 hours is 0.95. if the constant failure rate is given by λ, we have

$$R(t) = e^{-\lambda t} \qquad or \quad 0.95 = e^{-\lambda(4000)}$$

This yields

$$\lambda = 0.0000128/\text{hour} = 12.8/10^6 \text{ hours}$$

Thus, the highest failure rate is $12.8/10^6$ hours for a reliability of 0.95 at 4000 hours.

- The automatic focus unit of a television camera has 10 components in series. Each component has an exponential time-to-failure distribution with a constant failure rate of 0.05 per 4000 hours. What is the reliability of each component after 2000 hours of operation? Find the reliability of the automatic focus unit for 2000 hours of operation. What is its mean time-to-failure?

Solution The failure rate for each component is

$$\lambda = 0.05/4000 \text{ hours} = 12.5 \times 10^{-6}/\text{hour}$$

The reliability of each component after 2000 hours of operation is

$$R = \exp[-(12.5 \times 10^{-6})2000] = 0.975$$

The reliability of the automatic focus unit after 2000 hours of operation is

$$R_s = \exp[-(10 \times 12.5 \times 10^{-6})2000] = 0.779$$

The mean time to failure of the automatic focus unit is

$$\text{MTTF} = 1/(10 \times 12.5 \times 10^{-6}) = 8000 \text{ hours}$$

Problem 1: The reliability of a cutting assembly is given by

$$R(t) = \begin{cases} (t - t/t_0)^2 & 0 \le t \le t_0 \\ 0 & t \ge t_0 \end{cases}$$

Determine (i) the failure rate
(ii) does failure rate increase or decrease with time
(iii) determine the MTTF

Solution:
From the relationship between f (t) and R (t), we have

$$(i) \qquad f(t) = \frac{d}{dt} R(t)$$

$$\therefore \qquad f(t) = \frac{-d}{dt}\left(1 - \frac{t}{t_0}\right)^2 = \frac{2}{t_0}\left(1 - \frac{t}{t_0}\right) 0 \le t \le t_0$$

and the failure rate can be determined by

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{2}{t_0(1 - t/t_0)} 0 \le t \le t_0$$

(ii) The failure rate increases from $2/t_0$ at $t = 0$ to ∞ at $t = t_0$.

(iii) Mean time to failure

$$\text{MTTF} = \int_0^\infty R(t)\,dt$$

$$= \int_0^\infty dt\,(1 - t/t_0)^2 = t_0/3$$

Problem 2: Pdf for a random variable T, the time in operating hours to failure of an engine is given. What is the reliability for 100 hr operating life.

$$f(t) = \begin{cases} \dfrac{0.001}{(0.001t+1)^2} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$R(t) = \int_t^\infty f(t)\,dt = \int_t^\infty \frac{0.001}{(0.001\,t+1)^2}\,dt$$

$$= \left.\frac{-1}{(0.001\,t^1+1)}\right|_t^\infty = \frac{1}{0.001t+1}$$

and

$$f(t) = 1 - R(t) = 1 - \frac{1}{0.001t+1} = \frac{0.001t}{0.001t+1}$$

then,

$$R(100) = \frac{1}{0.1+1} = 0.999.$$

Problem 3: *The probability density function is given by*

$$f(t) = \begin{cases} 0.002\, e^{-0.002t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with 't' hours. Determine R (t) and MTTF and also find the median time to failure.

Solution:

$$R(t) = \int_t^{\infty} 0.002\, e^{-0.002t}\, dt$$

$$= e^{-0.002\, t}$$

and

$$\text{MTTF} = \int_0^{\infty} e^{-0.002t}\, dt$$

$$= \left.\frac{-e^{0.002t}}{-0.002}\right|_0^{\infty} = \frac{1}{0.002} = 500 \text{ hrs.}$$

To find the median to failure

Put $R\,(t_{\text{med}}) = e^{-0.002\,t}\,\text{med} = 0.5$

Solving for $t_{\text{med}}, t_{\text{med}} = \dfrac{\ln 0.5}{-0.002} = 346.6 \text{ hrs.}$

Problem 4: *A particular machine has a constant failure rate of λ = 0.02 hrs.*

(a) What is the probability that it will fail within first 10 hours.

(b) Suppose that the machine has operated successfully operated for 100 hrs, what is the probability that it will fail during the next 10 hours of operation.

Solution: (*a*) Probability of failure for first 10 hours is

$$P\{t \le 10\} = \int_0^{10} f(t)dt = F(t)$$

$$= 1 - e^{-0.02 \times 10} = 0.181$$

(*b*) The conditional probability

$$P\{t \le 110 \,|\, , t > 100\} = P\left\{\frac{(t \le 100) \cap (t > 100)}{P(t > 100)}\right\}$$

$$= P\left\{\frac{(100 \le t > 100)}{P\{t > 100\}}\right\}$$

$$P\{t \le 100 \,|\, , t > 100\} = \int_{100}^{110} \frac{f(t)\,dt}{1 - F(100)}$$

$$= \int_{100}^{110} \frac{0.02\, e^{-0.02t}\,dt}{1 - 1t \exp(-0.02 \times 100)}$$

$$= \frac{\exp(-0.02 \times 100) - \exp(-0.02 \times 100)}{\exp(-0.02 \times 100)}$$

$$= 1 - \exp(-0.02 \times 100) = 0.181$$