

8 Applications of Correlation Process

9. Autocorrelation

10. Cross Correlation

11. Input-Output Correlation

12. Properties of Correlation Function



1. Representation of discrete time signals

There are three ways to represent discrete time signals.

1) Functional Representation

$$x(n) = \begin{cases} 4 & \text{for } n=1,3 \\ -2 & \text{for } n=2 \\ 0 & \text{elsewhere} \end{cases}$$

2) Tabular method of representation

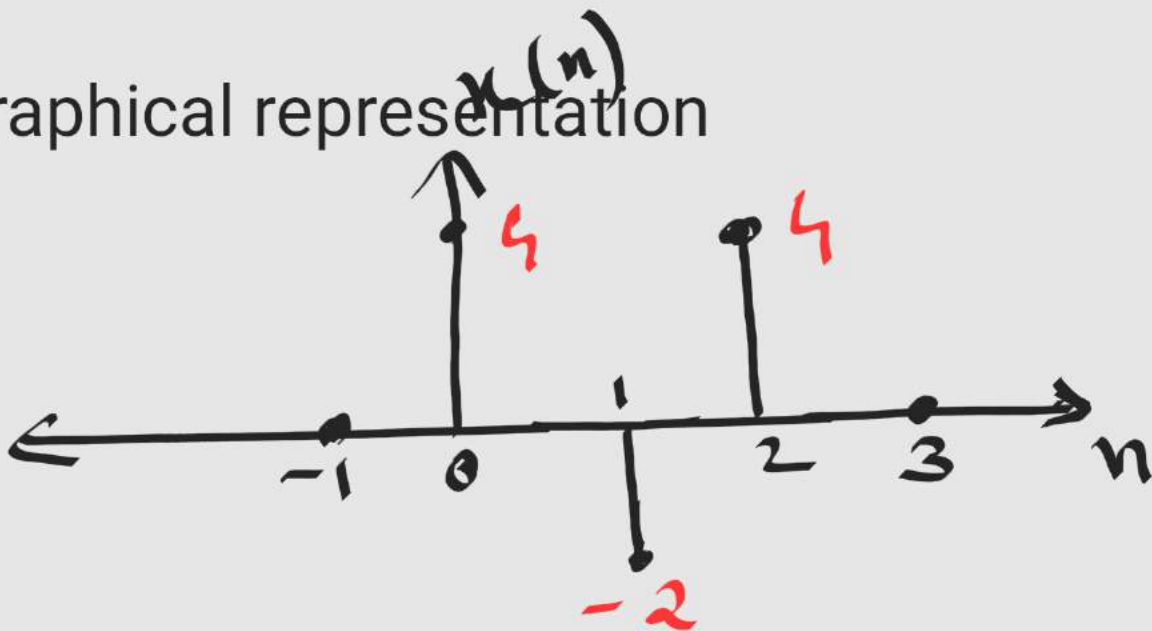
n	-3	-2	-1	0	1	2	3	4	5
x(n)	0	0	0	0	4	-2	4	0	0

$$X(n) = \{ 0, 4, -2, 4, 0, \dots \}$$

↑
n=0

3) Sequence Representation

4) Graphical representation



2. LTI System

Linear Time Invariant System

* Basic Knowledge – about System

What is System?

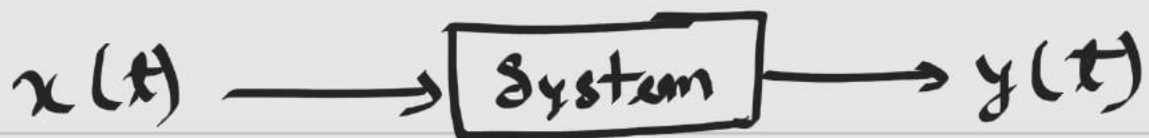
Types of System

Classification of CTS & DTS

Causality & Stability

Linear & Time Invariant

1) System - It may be defined as set of elements / functional blocks which are connected together & produces an o/p in response to an i/p signal.



→ The response of an system depends on T.F of a system

→ $x(t)$ is the i/p or excitation

→ $y(t)$ is the o/p or response

→ Means that i/p or excitation $x(t)$ produces an o/p or response $y(t)$

→ Examples of systems -

filters,
audio amplifiers,
communication
channel } Electrical
System

T.V set → interconnection of
(big system) small systems

↓
(tuner, IF amplifier)
Video amplifiers,
Sound "

2) Types of System →

Continuous time
System

Discrete
Time
System

3) Classification of systems →
(Check previous notes)

Linear Time Invariant System

LTI \rightarrow Linear (i)
 \rightarrow Time Invariant (ii) } Properties

1. What is LTI system?

\rightarrow If a system has both the linearity & time-invariance properties, then this system is called Linear-Time Invariant System.

2. Characterization of LTI system

(i) Linear System (Superposition theorem)

System to be linear —

Q: Eqn between y & x ,

Ex 1 \rightarrow $y(t) = tx(t)$

(1) For $x_1(t)$: $y(t) = tx_1(t)$
(i/p)

(2) For $x_2(t)$: $y(t) = tx_2(t)$

(3) $y'(t)$: $y'(t) = t[x_1'(t)] + t[x_2'(t)]$
(adding (1) & (2)) $= t[x_1(t) + x_2(t)]$

(4) $y''(t)$: $y''(t) = t[x_1(t) + x_2(t)]$
(Replace 't' with $x_1(t) + x_2(t)$)

$\therefore y'(t) = y''(t) \Rightarrow$

$\therefore \Rightarrow$ linear system

$\nexists \mathcal{I}_b$ $y'(t) \neq y''(t) \Rightarrow$ Non-linear

ii) Time Invariant System

① First Delay by k units

Delay $\Rightarrow x(t)$ by k

② Replace wherever you see ' t '
with $(t-k)$

Replace $\Rightarrow t \rightarrow (t-k)$

③ If step ① = step ② (TIV)

① \neq ② (TV)

TIV \rightarrow Time invariant

TV \rightarrow Time variant

Ex: $t x(t)$

① Delay: $y(t, k) = t x(t-k)$

② Replace: $y(t-k) = (t-k) x(t-k)$

③ $1 \neq 2 \Rightarrow$ Time variant //

Both CTS & DTS exhibit one property

→ Superposition Theorem can be

applied to find impulse response

$y(t)$ to a given function

✓ To find response of LTI system to

any given function, first we have

to find response of LTI system

For CTS

i/p → $x(t)$
 $\delta(t)$

o/p → $y(t)$

↓ $h(t)$

(impulse response)

for given $\delta(t)$

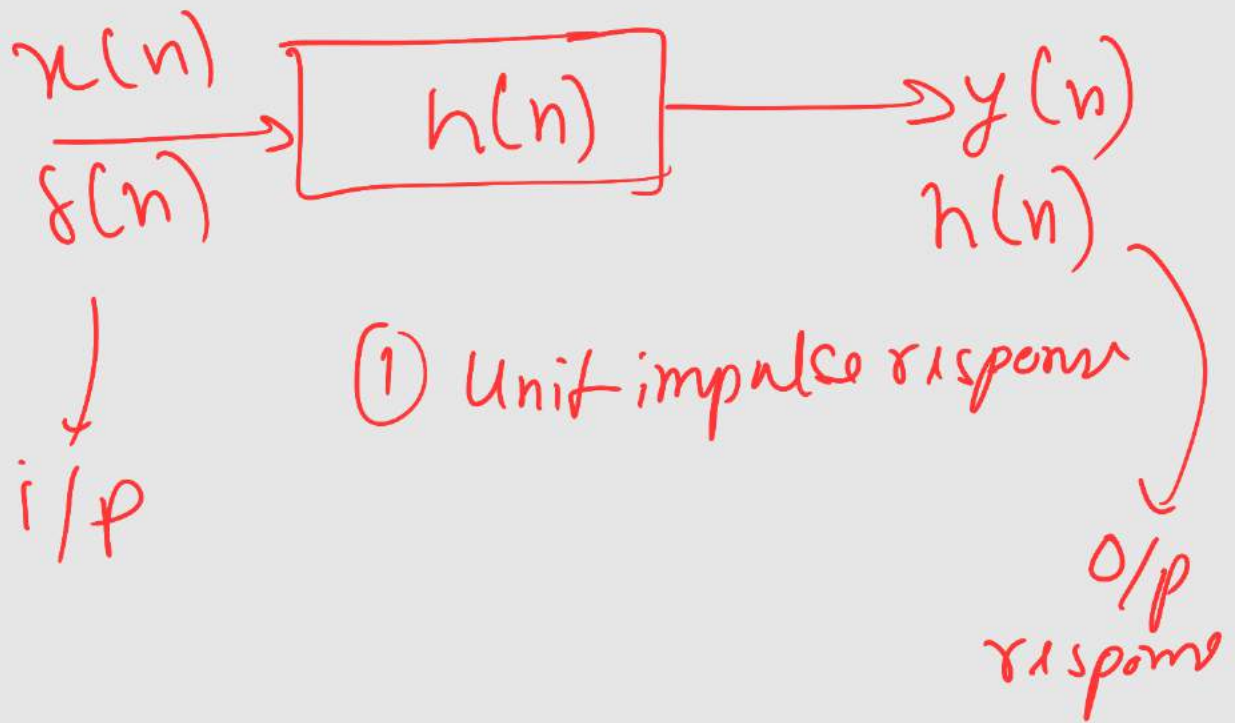
For DTS

i/p → $x(n)$
 $\delta(n)$

o/p → $y(n)$

$h(n)$

LTI System (Diagram)



CTS



DTs



3. Convolution Sum

→ Properties of LTI system

(output $y(n)$ in discrete LTI system can be expressed as a convolution sum)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

(or)

$$y(n) = x(n) * h(n)$$

Properties:

① Commutative

② Associative

③ Distributive

4. Static & Dynamic

5. Invertibility

✓ 6. Causality & Stability

2019

Convolution:

method
↑

Method 1: Conventional method

Ex: Find the convolution of two signals shown in fig ①

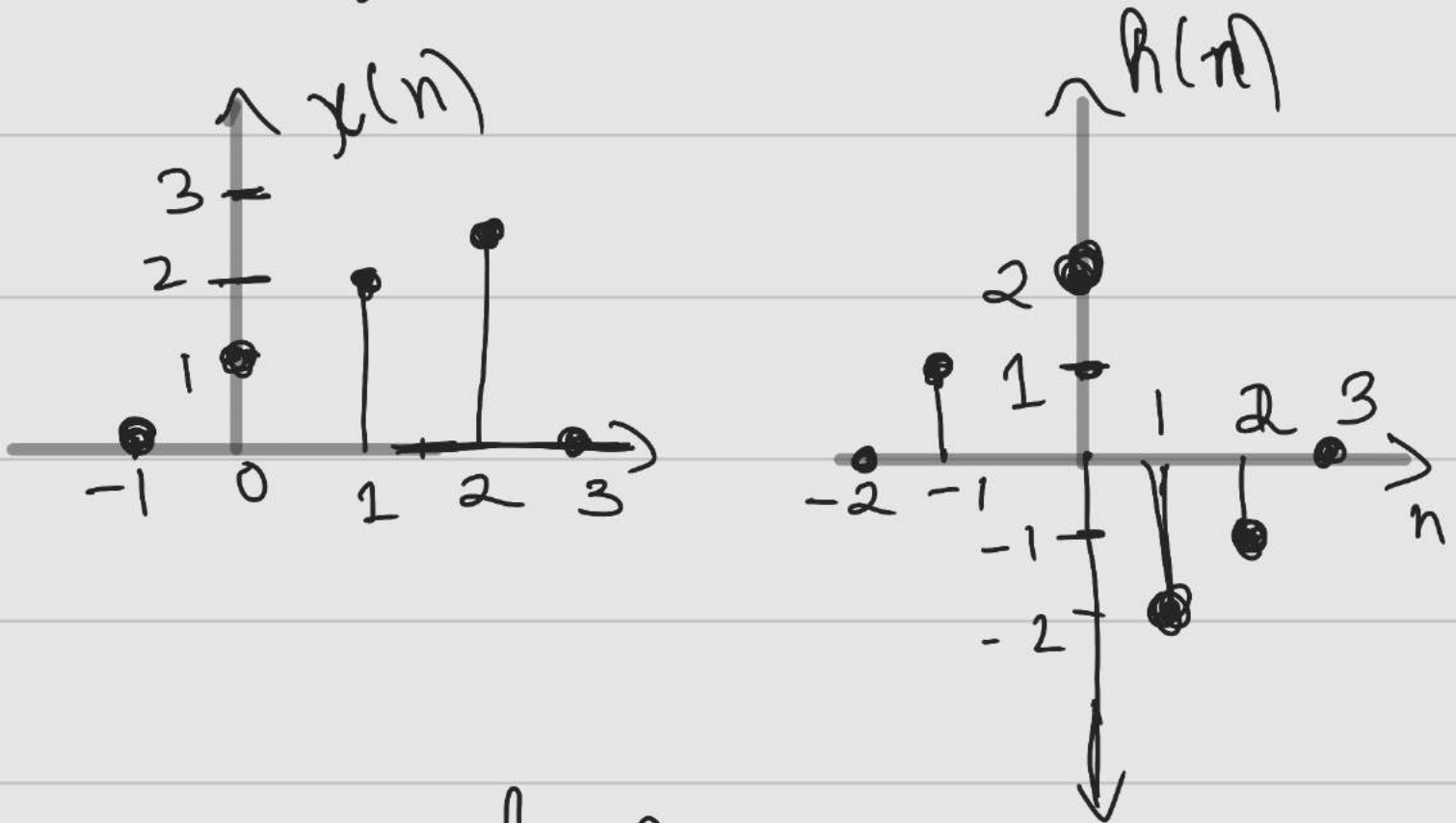


fig ①

Sol: Convolved signal

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

From graph.

$x(n)$ starts from 0 to 2

$h(n)$ starts from -1 to 2

$y(n)$ starts from $\underbrace{0 + (-1)}$ to $\underbrace{2 + 2}$
-1 to 4
 $n = \uparrow$ $n = \uparrow$

$R \rightarrow 0$ to 2

When $n = -1$

$$\begin{aligned}y(-1) &= \sum_{k=0}^2 x(k) h(-1-k) \\&= x(0) h(-1) + x(1) h(-2) \\&\quad x(2) h(-3) \\&= 1x1 + 2x0 + 2x0 \\&= 1\end{aligned}$$

When $n = 0$

$$y(0) = \sum_{k=0}^2 x(k) h(-k)$$

$$= x(0) h(0) + x(1) h(-1) + x(2) h(-2)$$

$$= 1x2 + 2x1 + 2x0 = 4$$

When $n = 1$

$$\underline{y(1)} = \sum_{k=0}^2 x(k) h(1-k)$$

$$= x(0)h(1) + x(1)h(0) + x(2)h(-1) \\ + x(3)h(-2)$$

$$= 1 \times -2 + 2 \times 2 + 3 \times 1$$

$$= 5$$

When $n = 2$

$$\underline{y(2)} = 1$$

When $n = 3$

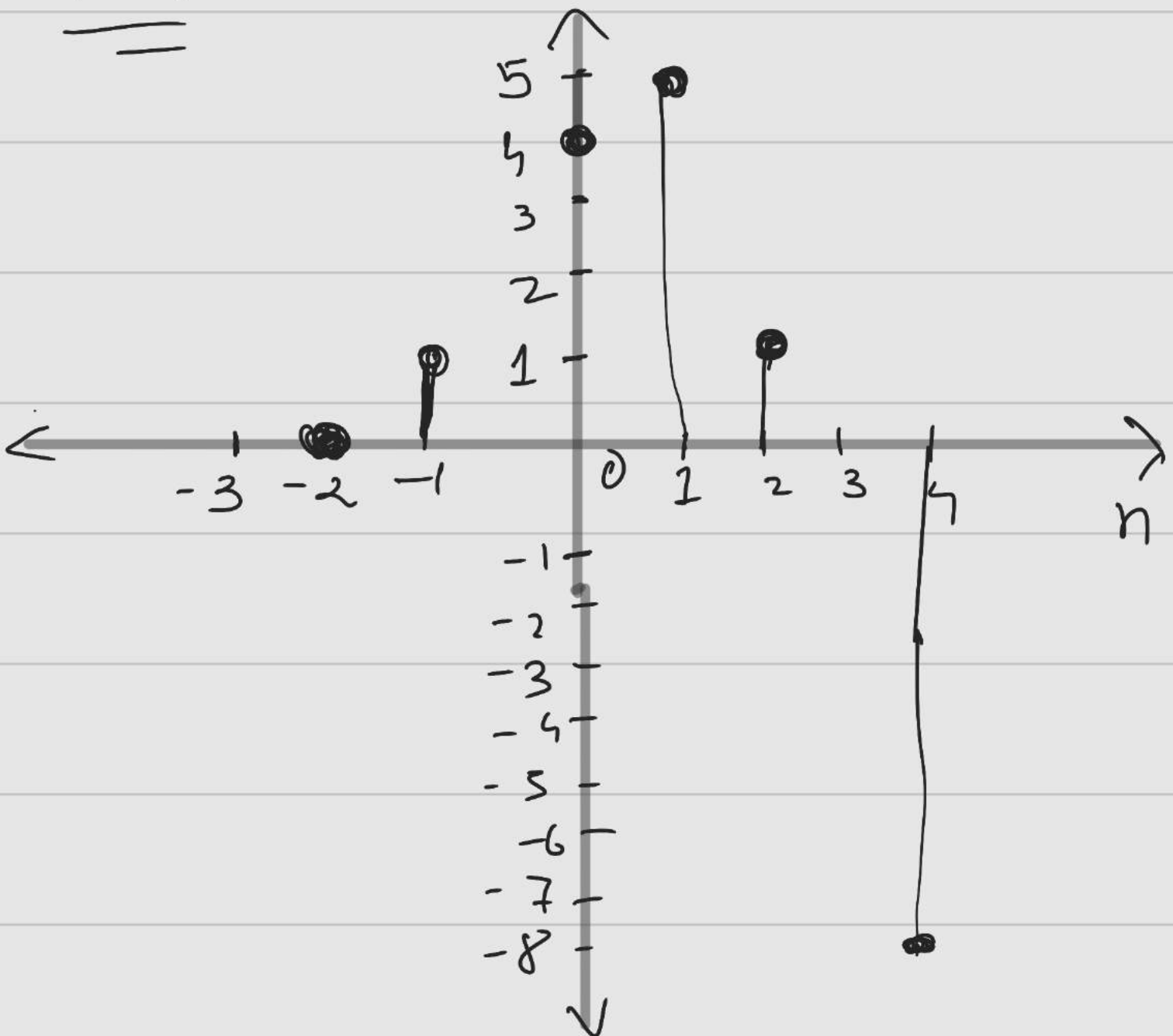
$$\underline{y(3)} = -8$$

When $n = 4$

$$\underline{y(4)} = -3$$

$$y[n] = [2, 4, 5, 1, -8, -3]$$

Plot :



Example 5.22 Find the convolution of two finite duration sequences

Conventional

$$x(n) = \begin{cases} 1, & -1 \leq n \leq 1 \\ 0, & \text{otherwise} \end{cases} \text{ and } h(n) = \begin{cases} 1, & -1 \leq n \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution The convolution of two finite duration sequences is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad \text{or} \quad y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

Step 1 Plot the given sequence, as shown in Fig. E5.22(a).

Step 2 To find the convolution sum $y(n)$

When $n = 0$

$$\begin{aligned} y(0) &= \sum_{k=-\infty}^{\infty} x(k)h(-k) \\ &= \dots + x(-1)h(1) + x(0)h(0) + x(1)h(-1) + \dots \\ &= 0 + (1)(1) + (1)(1) + (1)(1) + 0 \dots = 3 \end{aligned}$$

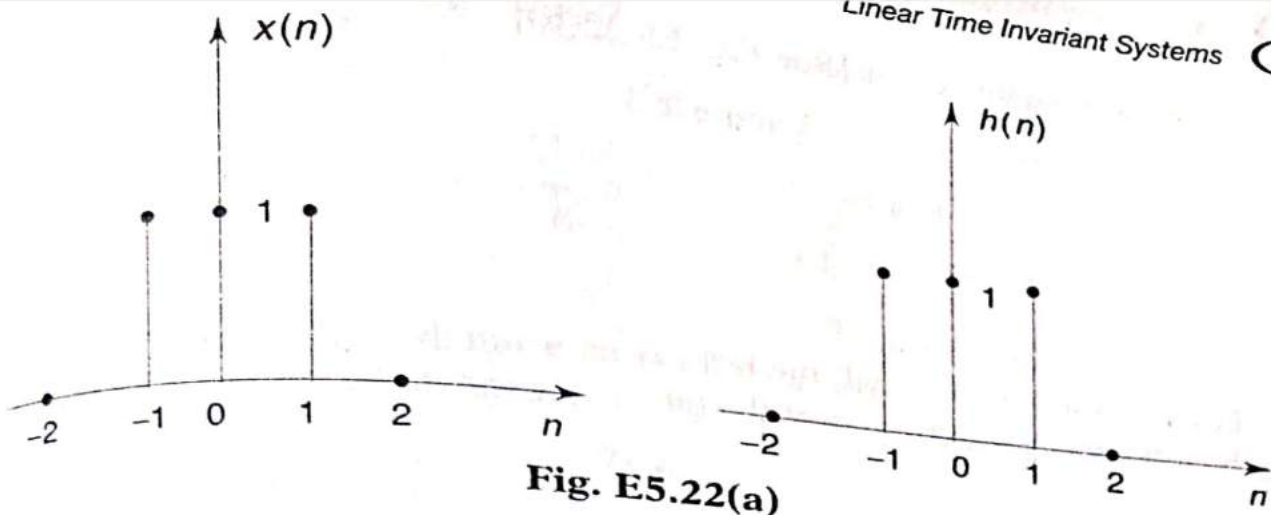


Fig. E5.22(a)

 When $n = 1$

$$\begin{aligned}
 y(1) &= \sum_{k=-\infty}^{\infty} x(k)h(1-k) \\
 &= \dots + x(-1)h(2) + x(0)h(1) + x(1)h(0) + \dots \\
 &= 0 + (1)(1) + (1)(1) + 0 \dots = 2
 \end{aligned}$$

 When $n = 2$

$$\begin{aligned}
 y(2) &= \sum_{k=-\infty}^{\infty} x(k)h(2-k) \\
 &= \dots + x(-1)h(3) + x(0)h(2) + x(1)h(1) + \dots \\
 &= 0 + (1)(1) + 0 \dots = 1
 \end{aligned}$$

 When $n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 0$$

 When $n = -1$

$$\begin{aligned}
 y(-1) &= \sum_{k=-\infty}^{\infty} x(k)h(-1-k) \\
 &= \dots + x(-1)h(1) + x(0)h(-1) + x(1)h(-2) + \dots \\
 &= 0 + (1)(1) + (1)(1) + 0 \dots = 2
 \end{aligned}$$

 When $n = -2$

$$\begin{aligned}
 y(-2) &= \sum_{k=-\infty}^{\infty} x(k)h(-2-k) \\
 &= \dots + x(-1)h(-1) + x(0)h(-2) + x(1)h(-3) + \dots \\
 &= 0 + (1)(1) + 0 \dots = 1
 \end{aligned}$$

 When $n = -3$

$$y(-3) = \sum_{k=-\infty}^{\infty} x(k)h(-3-k) = 0$$

The convolution signal $y(n)$ is [See Fig. E5.22(b)]

$$y(n) = 0, n \leq -3 \text{ and } n \geq 3$$

$$y(n) = 1, n = \pm 2$$

$$y(n) = 2, n = \pm 1$$

$$y(n) = 3, n = 0$$

Note: For the convolved signal, the left extreme and the right extreme can be found using the left and right extremes of the two sequences to be convolved. That is,

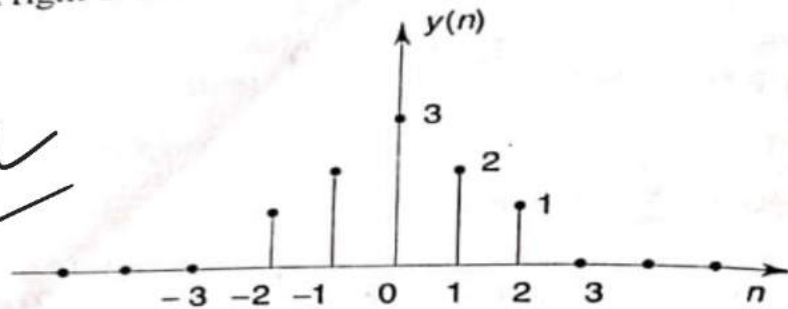


Fig. E5.22(b)

Method
Graphical

$$y_l = x_l + h_l$$

$$y_r = x_r + h_r$$

where x_l, h_l and y_l are the left extremes of the signals x, h and y respectively. Similarly x_r, h_r and y_r are the right extremes of the signals x, h and y respectively.

Alternate (Graphical) Method

The given problem can be solved by using the graphical method as shown in

Fig. E5.22(c). We know that $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

When $n = 0, y(0) = \sum_{k=-\infty}^{\infty} y_0(k) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 3$

When $n = 1, y(1) = \sum_{k=-\infty}^{\infty} y_1(k) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 2$

When $n = 2, y(2) = \sum_{k=-\infty}^{\infty} y_2(k) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 1$

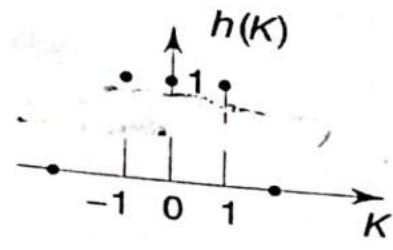
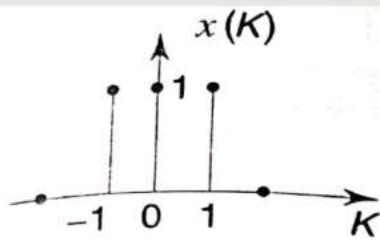
When $n = 3, y(3) = \sum_{k=-\infty}^{\infty} y_3(k) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 0$

When $n = -1, y(-1) = \sum_{k=-\infty}^{\infty} y_{-1}(k) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 2$

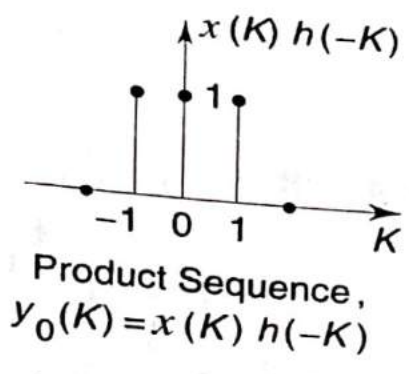
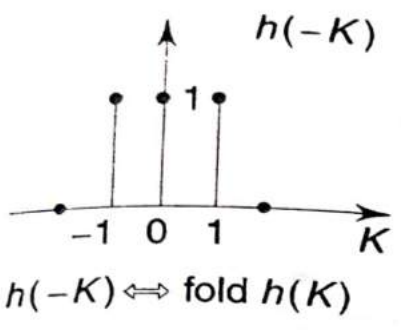
When $n = -2, y(-2) = \sum_{k=-\infty}^{\infty} y_{-2}(k) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k) = 1$

When $n = -3, y(-3) = \sum_{k=-\infty}^{\infty} y_{-3}(k) = \sum_{k=-\infty}^{\infty} x(k)h(-3-k) = 0$

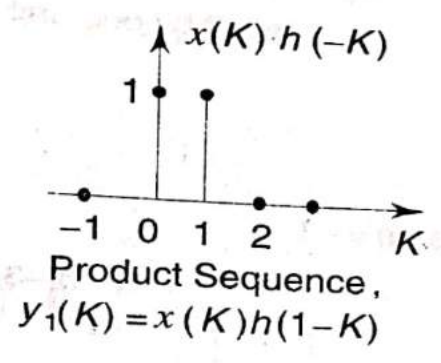
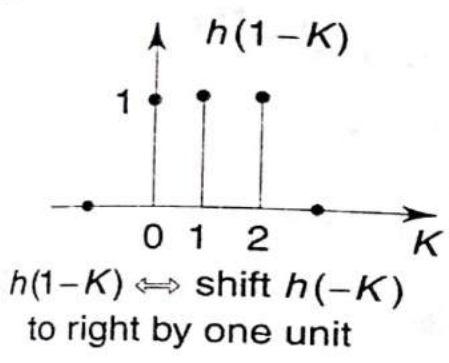
When these sequence values are plotted in Fig. E5.22(c), we find that the result is identical to the result shown in Fig. E5.22(b).



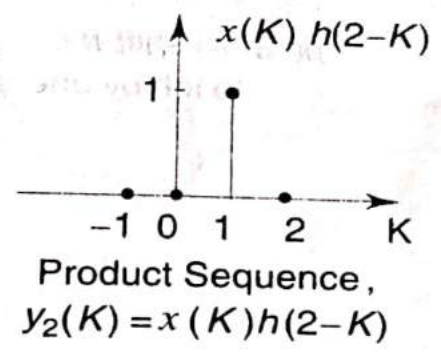
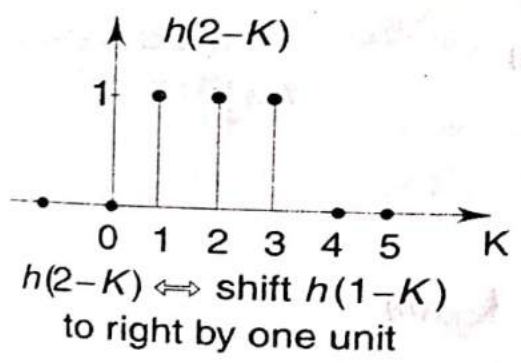
When $n = 0$



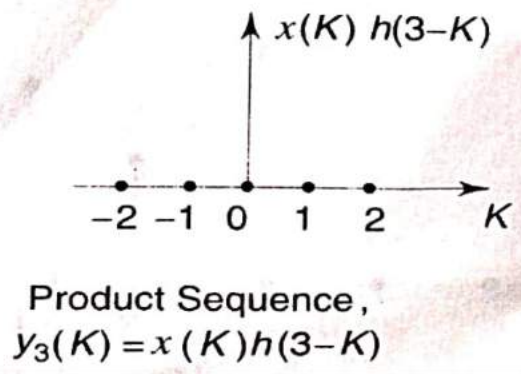
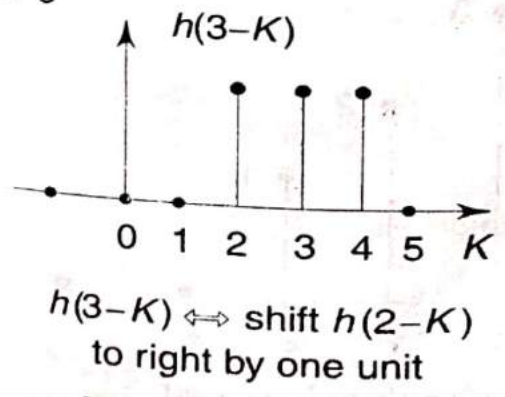
When $n = 1$



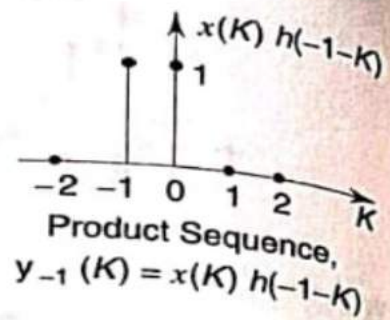
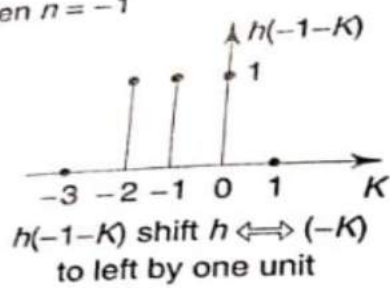
When $n = 2$



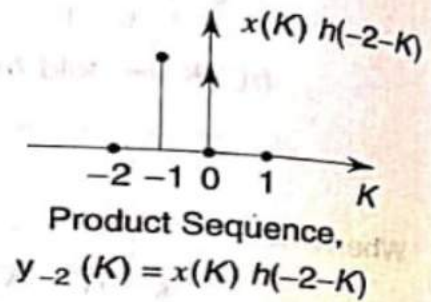
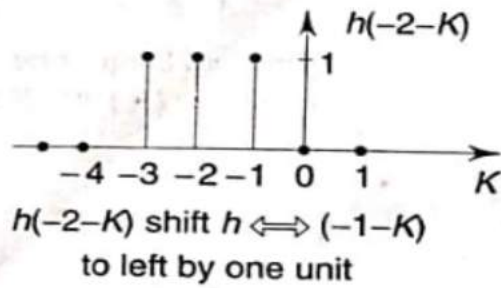
When $n = 3$



When $n = -1$



When $n = -2$



When $n = -3$

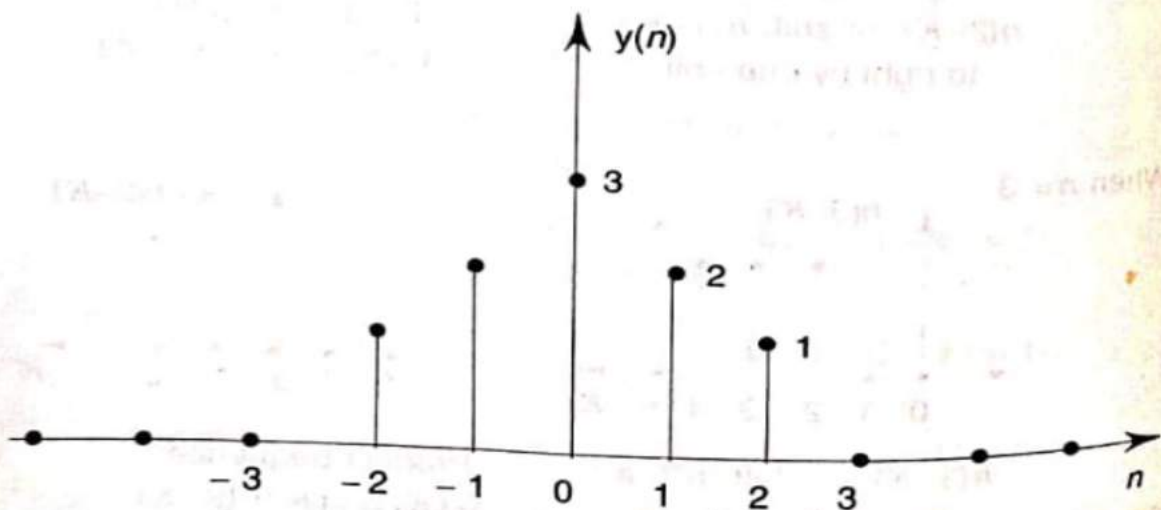
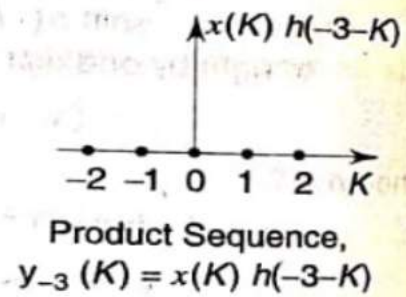
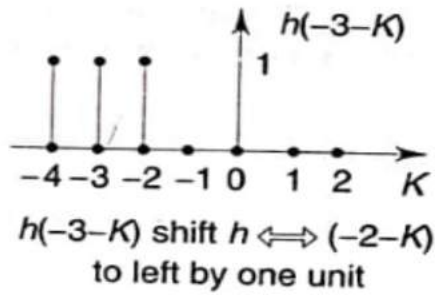


Fig. E6.3(c) Graphical Method

These

Example 5.25 Compute the convolution $y(n) = x(n) * h(n)$ of the signals

$$x(n) = \left\{ \begin{matrix} 1, 1, 0, 1, 1 \\ \uparrow \end{matrix} \right\} \text{ and } h(n) = \left\{ \begin{matrix} 1, -2, -3, 4 \\ \uparrow \end{matrix} \right\}$$

Solution The sequences of the given two signals are plotted in Fig. E 5.25(a).

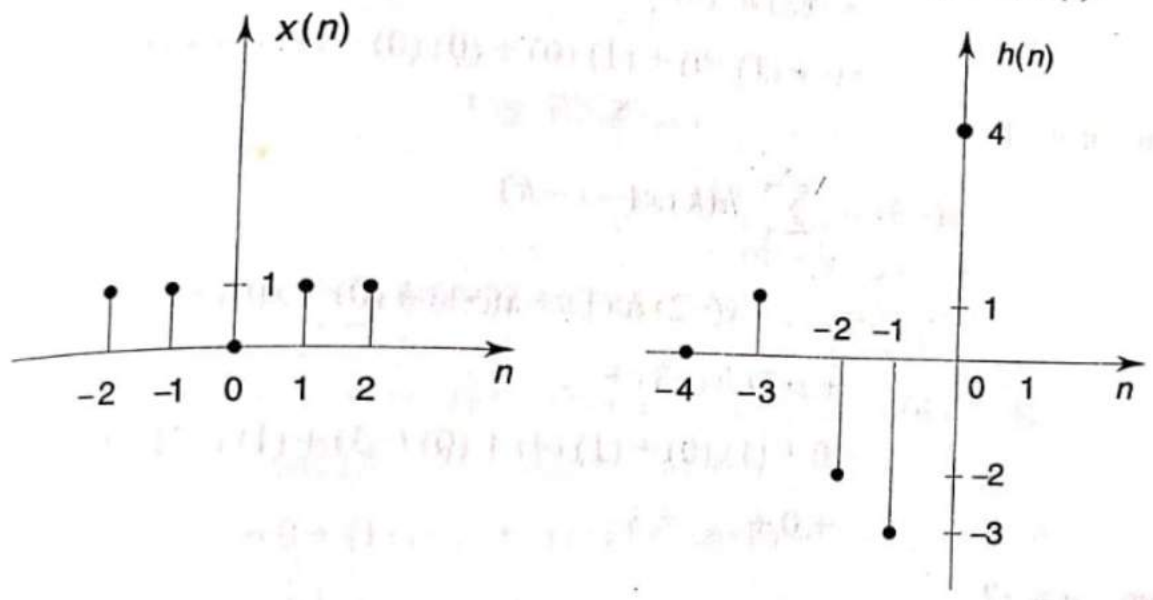


Fig. E5.25(a)

From the graph,

$$x_l = -2, x_r = 2, h_l = -3, h_r = 0$$

Hence the left and right extremes of the convoluted signal $y(n)$ are calculated as

$$y_l = x_l + h_l = -2 + (-3) = -5$$

$$y_r = x_r + h_r = 2 + 0 = 2$$

The convolution signal $y(n)$ is given as

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

When $n=0$

$$\begin{aligned} y(0) &= \sum_{k=-\infty}^{\infty} h(k)x(-k) \\ &= \dots + 0 + x(-2)h(2) + x(-1)h(1) + x(0)h(0) + x(1) \\ &\quad h(-1) + x(2)h(-2) + 0 + \dots \end{aligned}$$

$$= 0 + (1)(0) + (1)(0) + (0)(4) + (1)(-3) + (1)(-2) + 0 + \dots = -5$$

When $n = 1$

$$\begin{aligned} y(1) &= \sum_{k=-\infty}^{\infty} h(k)x(1-k) \\ &= \dots + x(-2)h(3) + x(-1)h(2) + x(0)h(1) + x(1)h(0) \\ &\quad + x(2)h(-1) + \dots \\ &= 0 + (1)(0) + (1)(0) + (0)(0) + (1)(4) + (1)(-3) + 0 + \dots = 1 \end{aligned}$$

When $n = 2$

$$\begin{aligned} y(2) &= \sum_{k=-\infty}^{\infty} h(k)x(2-k) \\ &= \dots + x(-2)h(4) + x(-1)h(3) + x(0)h(2) + x(1)h(1) \\ &\quad + x(2)h(0) + \dots \\ &= 0 + (1)(0) + (1)(0) + (0)(0) + (1)(4) + (1)(0) + \dots = 4 \end{aligned}$$

When $n = -1$

$$\begin{aligned} y(-1) &= \sum_{k=-\infty}^{\infty} h(k)x(-1-k) \\ &= \dots + x(-2)h(1) + x(-1)h(0) + x(0)h(-1) + x(1)h(-2) \\ &\quad + x(2)h(-3) + \dots \\ &= 0 + (1)(0) + (1)(4) + (0)(-3) + (1)(-2) + (1)(1) \\ &\quad + 0 + \dots = 3 \end{aligned}$$

When $n = -2$

$$\begin{aligned} y(-2) &= \sum_{k=-\infty}^{\infty} h(k)x(-2-k) \\ &= \dots + x(-2)h(0) + x(-1)h(-1) + x(0)h(-2) + \\ &\quad x(1)h(-3) + x(2)h(-4) + \dots \\ &= 0 + (1)(4) + (1)(-3) + (0)(-2) + (1)(1) + (1)(0) \\ &\quad + 0 \dots = 2 \end{aligned}$$

When $n = -3$

$$\begin{aligned} y(-3) &= \sum_{k=-\infty}^{\infty} h(k)x(-3-k) \\ &= \dots + x(-2)h(-1) + x(-1)h(-2) + x(0)h(-3) + \\ &\quad x(1)h(-4) + x(2)h(-5) + \dots \\ &= 0 + (1)(-3) + (1)(-2) + (0)(1) + (1)(0) + (1)(0) \\ &\quad + 0 \dots = -5 \end{aligned}$$

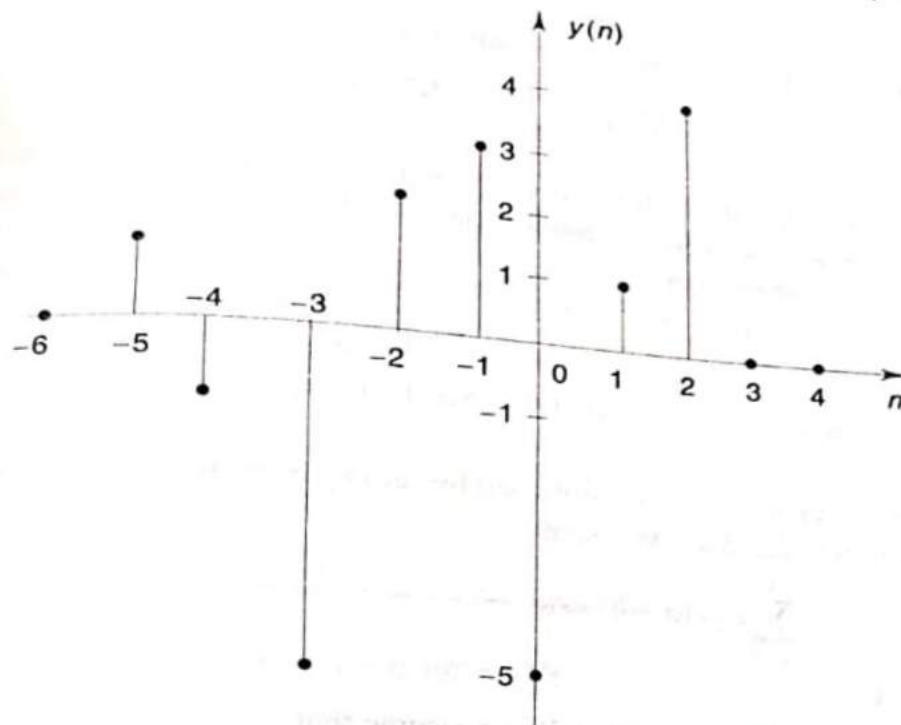


Fig. E5.25(b)

 When $n = -4$

$$\begin{aligned}
 y(-4) &= \sum_{k=-\infty}^{\infty} h(k)x(-4-k) \\
 &= \dots + x(-2)h(-2) + x(-1)h(-3) + x(0)h(-4) + \\
 &\quad x(1)h(-5) + x(2)h(-6) + \dots \\
 &= 0 + (1)(-2) + (1)(1) + (0)(0) + (1)(0) + (1)(0) \\
 &\quad + 0 + \dots = -1
 \end{aligned}$$

 When $n = -5$

$$\begin{aligned}
 y(-5) &= \sum_{k=-\infty}^{\infty} h(k)x(-5-k) \\
 &= \dots + x(-2)h(-3) + x(-1)h(-4) + x(0)h(-5) + \\
 &\quad x(1)h(-6) + x(2)h(-7) + \dots \\
 &= 0 + (1)(1) + 0 + \dots = 1
 \end{aligned}$$

These sequence values are plotted in Fig. E5.25(b).

LINEAR CONSTANT

Convolution using

③ Tabular Method 3

Ex: $x(n) = \{-2, -1, 0, 1, 2\}$

$h(n) = \{-3, -2, -1, 0\}$

Sol:

$x(n) \backslash h(n)$	\downarrow -3	-2	-1	0
-2	+6	4	2	0
-1	3	2	-1	0
0	0	0	0	0
1	-3	-2	-1	0
2	-6	-4	-2	0

$n=0$
origin
diagonal
sum

① Multiply
 $x(n)$ $h(n)$
 sequence

② Add the sequence

$$y(n) = \left\{ \begin{array}{l} 1+6, 3+4, 0+2+2, -3+0, 1+0, \\ -6-2+0+0, -4-1+0, \\ -2+0, 0 \end{array} \right\}$$

$$y(n) = \left\{ 1+6, 7, 4, -2, -8, -2, 0 \right\}$$

↑
n=0

Ans

⇒ array method

7.7.5 Cross Correlation of Discrete-time Signals

(GTU, Gujrat, Sem. Exam., 2009-10)

The cross correlation of two discrete-time signals is a measure of similarity between them. We will define the cross correlation for energy and power discrete-time signals.

7.7.6 Cross Correlation of Discrete-time Energy Signals

Definition

Let x_1 and $x_2(n)$ denote a pair of real valued discrete-time energy signals. Then the cross correlation function of such a pair is defined as under:

$$R_{12}(k) = \sum_{n=-\infty}^N x_1(n) \cdot x_2(n-k) \quad \dots(7.70)$$

The second cross correlation function of $x_1(n)$ and $x_2(n)$ is defined as under:

$$R_{21}(k) = \sum_{n=-\infty}^{\infty} x_1(n-k) x_2(n) \quad \dots(7.71)$$

EXAMPLE 7.25. Obtain the cross correlation of the following sequences.

$$x_1(n) = [2, 3, 4] \text{ and } x_2(n) = [1, 2, 3]$$

Solution: There are two methods of solving this example. There are

1. Direct computation method
2. Graphical method

Method I: Direct computation method

The cross correlation is defined as under:

$$R_{12}(k) = \sum_{n=-\infty}^{\infty} x_1(n) x_2(n-k)$$

Now, let us obtain the values of $R_{12}(k)$ for different values of k . Since, $x_1(n)$ exists from $n = 0$ to $n = 2$, the product term $x_1(n) x_2(n-k)$ will be non-zero only for these values of n .

$$\text{Therefore, } R_{12}(k) = \sum_{n=0}^2 x_1(n) x_2(n-k)$$

1. $R_{12}(k)$ for $k = -3$.

Let us arbitrarily start at $k = -3$.

$$\begin{aligned} \text{Therefore, } R_{12}(-3) &= \sum_{n=0}^2 x_1(n) x_2(n+3) \\ &= x_1(0) x_2(3) + x_1(1) x_2(4) + x_1(2) x_2(5) \end{aligned}$$

$k \Rightarrow -3$ to 3

$$= (2 \times 0) + (3 \times 0) + (4 \times 0) = 0$$

This shows that for $k \leq -3$, $R_{12}(k) = 0$

2. $R_{12}(k)$ for $k = -2$.

$$\begin{aligned} \text{We have } R_{12}(-2) &= \sum_{n=0}^2 x_1(n) x_2(n+2) \\ &= x_1(0) x_2(2) + x_1(1) x_2(3) + x_1(2) x_2(4) \\ &= (2 \times 3) + (3 \times 0) + (4 \times 0) = 6 \end{aligned}$$

3. $R_{12}(k)$ for $k = -1$.

$$\begin{aligned} \text{We have } R_{12}(-1) &= \sum_{n=0}^2 x_1(n) x_2(n+1) \\ &= x_1(0) x_2(1) + x_1(1) x_2(2) + x_1(2) x_2(3) \\ &= (2 \times 2) + (3 \times 3) + (4 \times 0) = 13 \end{aligned}$$

4. $R_{12}(k)$ for $k = 0$.

$$\begin{aligned} \text{We have } R_{12}(0) &= \sum_{n=0}^2 x_1(n) x_2(n) \\ &= x_1(0) x_2(0) + x_1(1) x_2(1) + x_1(2) x_2(2) \\ &= (2 \times 1) + (3 \times 2) + (4 \times 3) = 20 \end{aligned}$$

5. $R_{12}(k)$ for $k = 1$.

$$\begin{aligned} \text{We have } R_{12}(1) &= \sum_{n=0}^2 x_1(n) x_2(n-1) \\ &= x_1(0) x_2(-1) + x_1(1) x_2(0) + x_1(2) x_2(1) \\ &= (2 \times 0) + (3 \times 1) + (4 \times 2) = 11 \end{aligned}$$

6. $R_{12}(k)$ for $k = 2$.

$$\begin{aligned} \text{We have } R_{12}(2) &= \sum_{n=0}^2 x_1(n) x_2(n-2) \\ &= x_1(0) x_2(-2) + x_1(1) x_2(-1) + x_1(2) x_2(0) \\ &= (2 \times 0) + (3 \times 0) + (4 \times 1) = 4 \end{aligned}$$

7. $R_{12}(k)$ for $k = 3$.

$$\begin{aligned} R_{12}(3) &= \sum_{n=0}^2 x_1(n) x_2(n-3) \\ &= x_1(0) x_2(-3) + x_1(1) x_2(-2) + x_1(2) x_2(-1) \\ &= (2 \times 0) + (3 \times 0) + (4 \times 0) = 0 \end{aligned}$$

The value of $R_{12}(k) = 0$ for $k \geq 3$

Plot of $R_{12}(k)$.

Figure 7.22 shows the plot of $R_{12}(k)$ with respect to k .

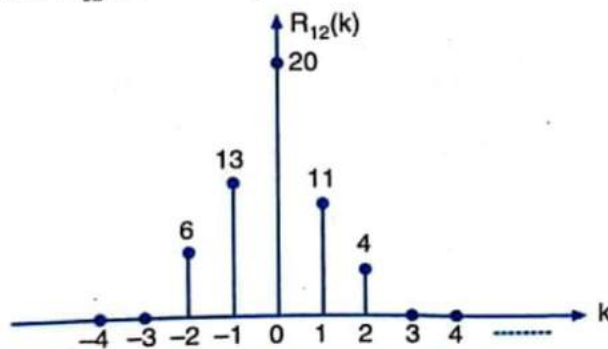


Fig. 7.22. Plot of $R_{12}(k)$

$R_{12}(k)$ in the sequence form.

$R_{12}(k)$ in the sequence form is as under:

$$R_{12}(k) = (6, 13, 20, 11, 4)$$

↑

(Online exam) shortcut

Obtain cross-correlation between, (Tabular method)

$$x(n) = [0, 1, 2, 3]$$

$$y(n) = [1, 2, 2, 1]$$

Sol: $y(-n) = [1, 2, 2, 1]$

$x(n) \backslash y(-n)$	1	2	1	1
0	0	0	0	0
1	1	2	1	1
2	2	4	2	2
3	3	6	3	1

Answer

$$R_{xy}(n) = \{0, 1, 4, 8, 9, 5, 1\}$$