Bineswar Brahma Engineering College

Mechanical Engineering Department

Dr. Mantulal Basumatary

Assistant Professor

Module-1: Equilibrium of Rigid Bodies

Engineering Mechanics: ME181104

Introduction: A rigid body is a body which does not deform or the deformation is negligible under the influence of external force or couple. A body or particle is said to be in equilibrium if it is at rest or has constant velocity if originally in motion. The term static equilibrium is used to describe an object at rest. The condition of equilibrium is obtained when the resultant force and couple acting on a body are equal to zero. Mathematically, the condition of equilibrium is,

$$\sum \boldsymbol{F} = 0 \text{ and } \sum \boldsymbol{M}_A = 0 \tag{1}$$

where $\sum \boldsymbol{F}$ and $\sum \boldsymbol{M}_A$ are the vector sum of all forces and moments acting on the particle and A is any point on the body about which moment is taken. If m and $\boldsymbol{a}$ represent mass and acceleration of the particle, from Newton's second law we know,

$$\sum \boldsymbol{F} = m\boldsymbol{a} \tag{2}$$

Using $\sum \boldsymbol{F} = 0$ in the above equation, we get

$$\boldsymbol{a} = 0 \qquad [\because m \neq 0] \tag{3}$$

Consequently, from equation (3) we can conclude that the particle indeed moves with a constant velocity or remains at rest when it is in equilibrium in the absence of a couple.

Free Body Diagram (*FBD*): We must account for all known and unknown forces acting on a particle/body to apply the equation of equilibrium. The best way to do this is to draw the particles' *free body diagram*. This diagram is simply a sketch which shows the particle free from its surroundings with all the forces (known or unknown) acting on it. The following three steps are essential to draw a free body diagram:

1. *Draw outlined shape*: Imagine the body to be isolated or cut free from its surroundings by drawing its outlined shape.

2. *Show all forces*: Indicate all the forces on this sketch which are acting on the body. These forces can be *active* forces, which tend to set the body in motion, or they can be *reactive* forces which are the result of the constraints or supports that tend to prevent the motion. To account for all these forces, it may help to trace around the body's boundary, carefully noting each force acting on it.

3. *Identify each force*: The forces that are known should be labeled with their proper magnitudes and directions

For example, Figure 1 shows the free body diagram of a rectangular block on an inclined plane showing the various forces acting on the rectangular block.

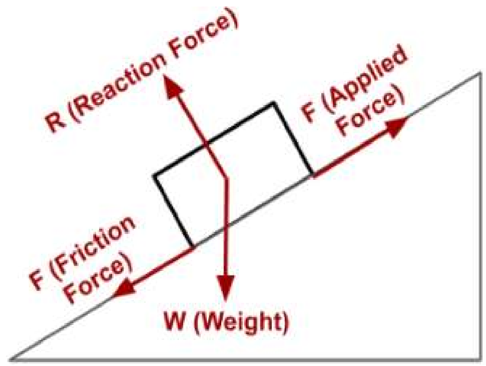


Figure 1: Free body diagram for a rectangular block moving on an inclined plane.

Free body diagram is a very important concept in mechanical engineering to solve static and dynamic problems. In conclusion, free body diagram can be used for (a) visualizing forces acting on a moving or stationary body, and (b) calculating unknown forces acting on a body.

Types of supports and their reactions: The reactions exerted on a two-dimensional structure can be divided into three groups corresponding to three types of supports or connections (see Figure 2):

1. *Reactions equivalent to a force with known line of action*: Supports and connections causing this type of reactions are rollers, rockers, frictionless surfaces, short links and cables, collars on frictionless rods, and frictionless pins in slots. Each of these supports and connections can prevent motion in one direction only. Each of these reactions involves one unknown, namely, the magnitude of the reaction. The line of action of the reaction is known and should be indicated clearly in the free-body diagram. The reaction can be directed either way in the case of double track rollers, links, collars on rods, and pins in slots. Single track rollers and rockers are generally assumed to be reversible, and thus the corresponding reactions can also be directed either way.

2. *Reactions equivalent to a force of unknown direction and magnitude*: Supports and connections cause these types of reactions include frictionless pins in fitted holes, hinges, and rough surfaces. They prevent translation of the free body in all directions, but they cannot prevent the body from rotating about the connection. Reactions of this group involve *two unknowns* and are usually

represented by their x and y components. In the case of a rough surface, the component normal to the surface must be directed away from the surface.
3. *Reactions equivalent to a force and a couple*: These reactions are caused by fixed supports which oppose any motion of the free body and thus constrain it completely. Reactions of this group involve *three unknowns,* consisting usually of the two components of the force and the moment of the couple.

System of Forces: Before going to discuss about force system, we must know what force is. Force is a quantity which can change the position of a body at rest or in motion or keep it in equilibrium. Force is a vector quantity that is it has direction as well as magnitude. The unit of force is Newton (N). Force can be inward/push i.e. directed towards the body and outward/pull, the direction of the force is away from the body. These two natures of force are known as compressive and tensile force.

The force system is of two types: (a) *coplanar force system*, and (b) *non-coplanar force system*. A force system is called as coplanar force system when all the forces lie on the same plane. Conversely, a force system is known as non-coplanar force system when all forces do not lie on the same plane. The coplanar force system can be further categorized as follows:

- *Collinear force system*: When all the forces lie in a straight line then that force system is called a collinear force system.
- *Concurrent force system*: When all the forces pass through a single point then that force system is called a concurrent force system.
- *Non-concurrent force system*: When all the forces pass through different points then that force system is known as non-concurrent force system.
- *Parallel force system*: When all the forces are parallel to each other, then that force system is known as parallel force system. In a parallel force system when all the forces act along the same direction, is called like parallel force system. On the other hand, in a parallel force system when all the forces do not direct along the same direction is known as unlike parallel system.

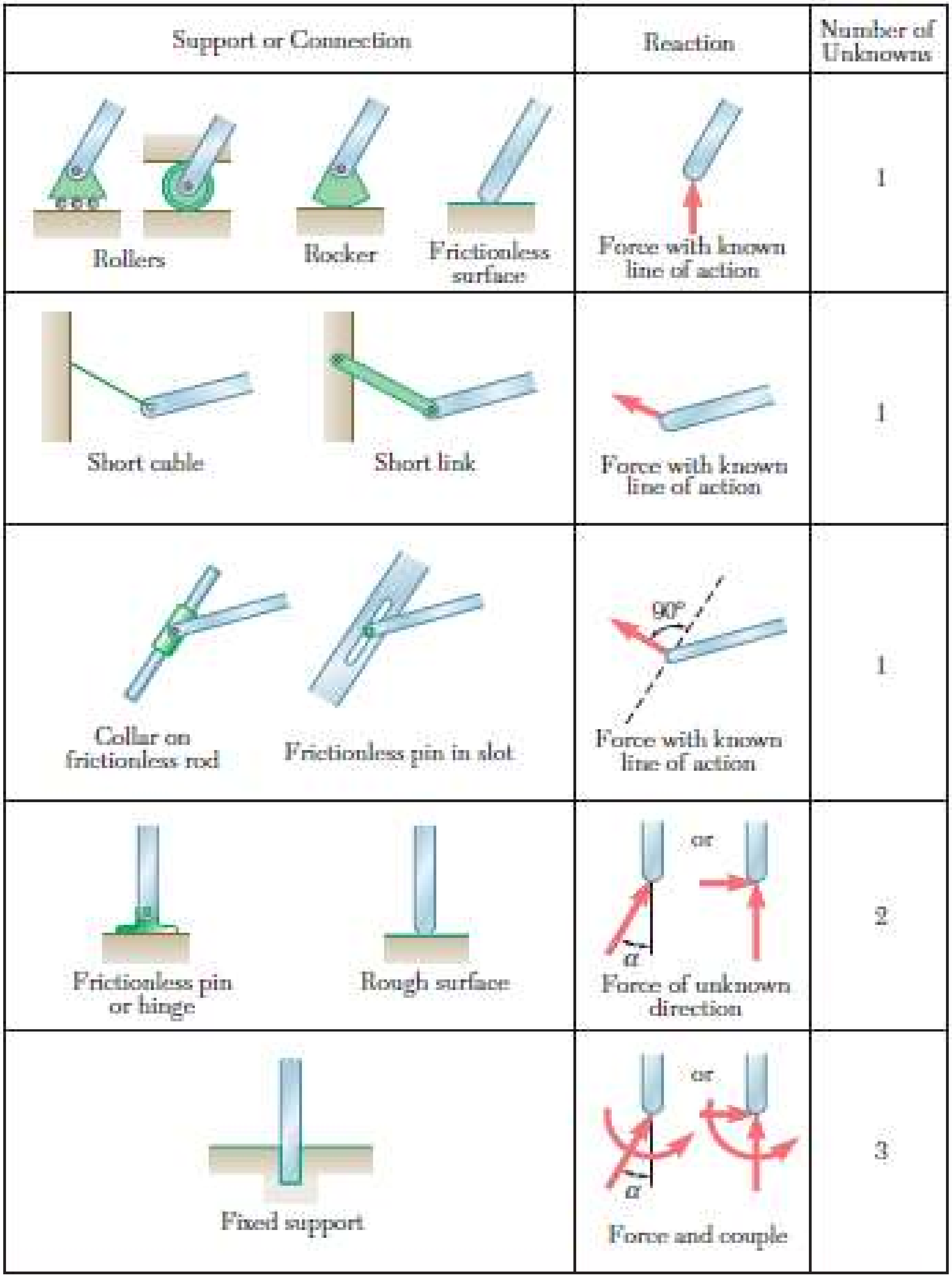


Figure 2: Reactions at supports and connections in two-dimensional structure.

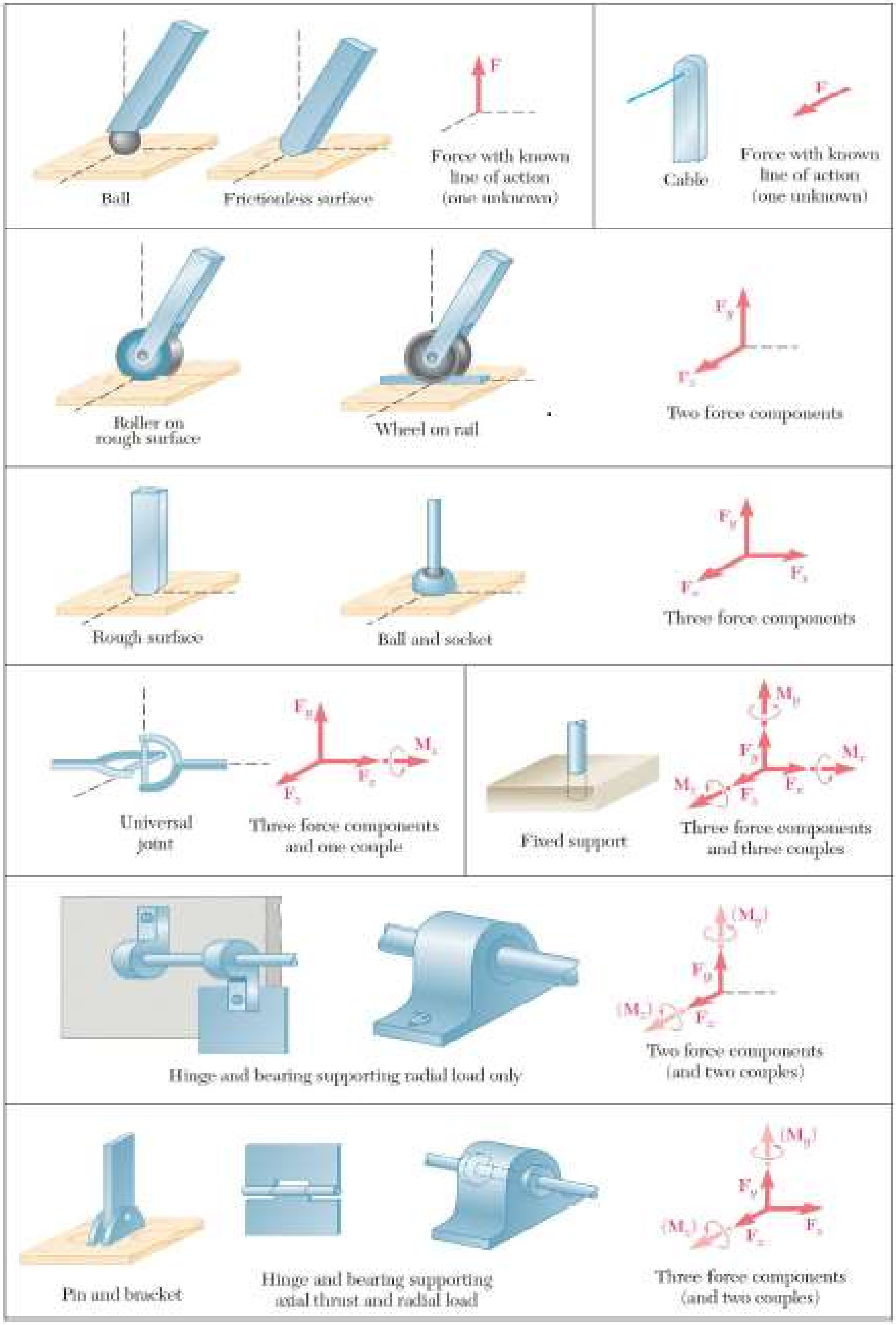


Figure 3: Reactions at supports and connections in three-dimensional structure.

Resultant of Coplanar Non-concurrent Force System: The resultant force of a coplanar non-concurrent force system can be found out analytically by the *method of resolution* of forces.

Method of resolution for the resultant force: To find out the resultant force by the resolution of forces the following steps can be followed:

1. Resolve all the forces horizontally and find the algebraic sum of all the horizontal components ($\sum H$).
2. Resolve all the forces vertically and find the algebraic sum of all the vertical components ($\sum V$).
3. The magnitude of the resultant R of the given forces will be given by

$$R = \sqrt{\left(\sum H\right)^2 + \left(\sum V\right)^2} \tag{4}$$

4. Direction of the resultant force: The resultant force will make an angle α with the horizontal will be given by

$$\tan \alpha = \frac{\sum V}{\sum H} \tag{5}$$

The value of the angle α will vary depending upon the sign of $\sum H$ and $\sum V$ as discussed below:

1) When the value of $\sum V$ is positive, the resultant makes an angle between 0° and 180° (i.e., the angle lies in the 1st or 2nd quadrant). But, when the value of $\sum V$ is negative, the resultant makes an angle between 180° and 360° (i.e., the angle lies in the 3rd or 4th quadrant).
2) When the value of $\sum V$ is positive, the resultant makes an angle between 0° and 90° or 270° to 360° (i.e., the angle lies in the 1st or 4th quadrant). But, when the value of $\sum H$ is negative, the resultant makes an angle between 90° and 270° (i.e., the angle lies in the 2nd or 3rd quadrant).

Resultant of Coplanar Concurrent Force System: For two coplanar concurrent forces, Parallelogram law of forces may be used to find the resultant force. Parallelogram law of forces states that *if two forces acting simultaneously on a particle, be represented in magnitude and direction by the two sides of a parallelogram, then their resultant force may be represented in magnitude and direction by the diagonal of the parallelogram which passes through their point* *of intersection*. Mathematically, magnitude of resultant force R of two forces F_1 and F_2 acting on a particle A as shown in Figure 4,

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta} \tag{6}$$

and

$$\tan\alpha = \frac{F_2 \sin\theta}{F_1 + F_2 \cos\theta} = \frac{\text{Vertical component of force}}{\text{Horizontal component of force}} \tag{7}$$

where θ is the angle between the forces F_1 and F_2, and α represent the angle which the resultant force R makes with the force F_1.

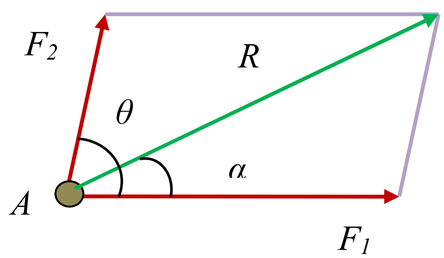


Figure 4: Finding out resultant force.

The process of splitting up the given force into a number of components, without changing its effect on the body is called resolution of a force. In a plane, a force is resolved generally along two mutually perpendicular directions. The principle of resolution states that *the algebraic sum of the resolved parts of a number of forces in a given direction is equal to the resolved part of their resultant in the same direction*. In general, in a plane the forces are resolved in the vertical (y-axis) and horizontal (x-axis) directions.

Note:

1) If $\theta = 0$, that is when the two forces act along the same line or direction, then $R = F_1 + F_2$
2) If $\theta = 90^\circ$, that is when the two forces act at right angle, then $R = \sqrt{F_1^2 + F_2^2}$
3) If $\theta = 180^\circ$, that is, when the forces act along the same line but in opposite direction, then $R = F_1 - F_2$ [$\because \cos 180^\circ = -1$]. In this case, the resultant force will act in the direction of the greater force.
4) If the two forces are equal in magnitude, it means $F_1 = F_2 = F$ (say), then $R = \sqrt{F^2 + F^2 + 2F^2\cos\theta} = \sqrt{2F^2(1+\cos\theta)} = \sqrt{2F^2 \times 2\cos^2\left(\frac{\theta}{2}\right)} = 2F\cos\left(\frac{\theta}{2}\right)$ [$\because 1 + \cos\theta = 2\cos^2(\theta/2)$].

Let us consider a particle *O* acted upon by several coplanar forces, i.e., by several forces contained in the same plane (see Figure 5(a)). Since the forces considered here all pass through *O*, so they are concurrent forces. The vectors representing forces acting on *O* may be added graphically by the polygon rule. Since the use of polygon rule is equivalent to the repeated application of the triangle law of forces (*If two forces acting simultaneously on a particle, be represented in magnitude and direction by the two sides of a triangle taken in order then the resultant force may be represented in magnitude and direction by the third side of the triangle taken in the opposite order.*), the vector *R* (see Figure 5(b)) thus obtained represents the resultant of the given concurrent forces F_1, F_2, and F_3, i.e., the single force which has the same effect on the particle *O* as the given forces. The order in which the given forces do not matter, but their direction should be kept the same.

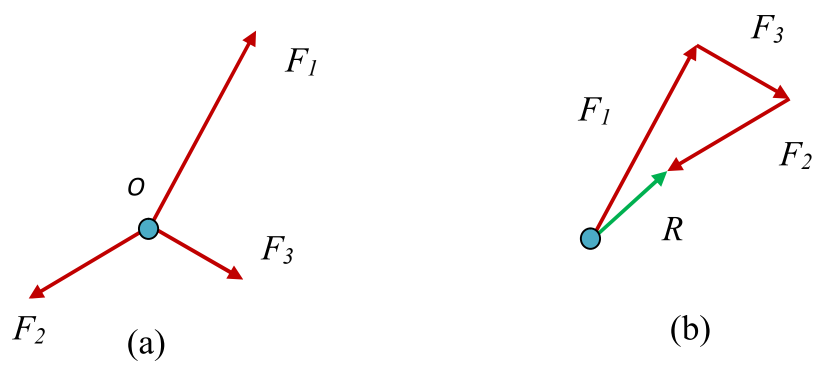


Figure 5: Coplanar concurrent forces acting on a particle and their resultant force by rule of polygon.

Lami's Theorem**:** This is an analytical method which can be used to calculate the resultant force of three coplanar concurrent forces acting on a body. It states that *if three coplanar forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two.* Mathematically,

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} \tag{8}$$

where, F_1, F_2, and F_3 represent the three forces acting opposite to the angles α, β and γ respectively as shown in the Figure 6.

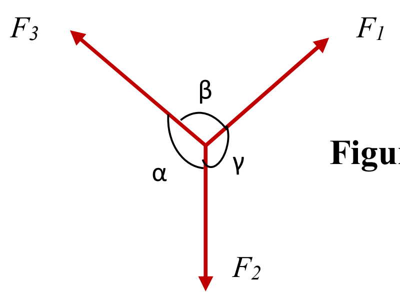


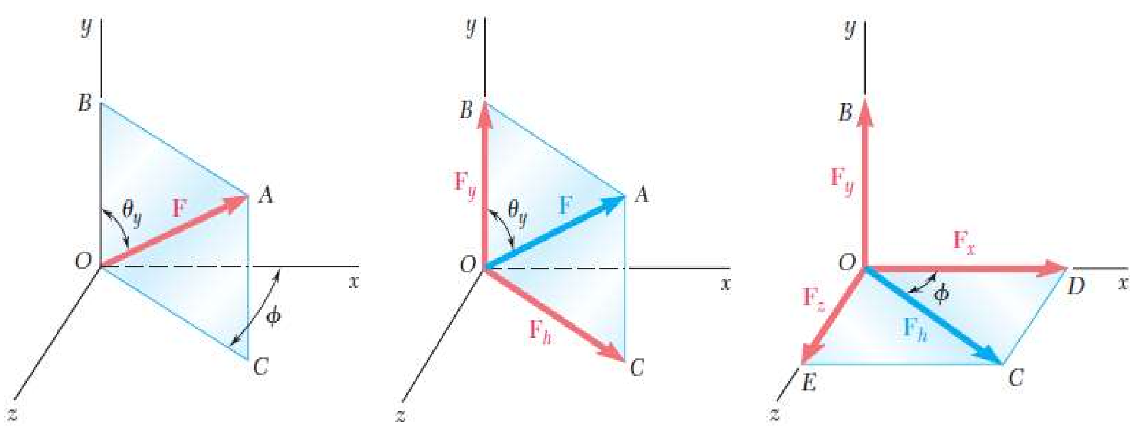
Figure 6: Lami's theorem for concurrent forces.

Non-concurrent Forces in Space: Resultant force of non-concurrent forces in space (three-dimension) can be found out using the *rectangular components* as shown in the Figure 7.

Let us consider a force ***F*** at the origin *O* of the system of rectangular coordinates *x, y,* and *z*. To define the direction of ***F*,** a vertical plane *OCAB* is drawn containing the force ***F*** as depicted in Figure 7(a) which passes through the vertical *y*-axis.

Let, the orientation of the plane *OCAB* is defined by the angle ϕ forming with the *xy*-plane (see Figure 7(a)). Let, the direction of ***F*** within the plane is defined by the angle θ_y that it forms with the vertical *y*-axis.

The force ***F*** may be resolved into a vertical component $\boldsymbol{F}_y$ and a horizontal component $\boldsymbol{F}_h$ in the plane *OCAB* as shown in the Figure 7(b). The components are given by,



(a) (b) (c)

Figure 7: Rectangular components of a force in space (three-dimension).

$F_y = |\boldsymbol{F}| cos\, \theta_y$, and $F_h = |\boldsymbol{F}|\, sin\, \theta_y$ (9)

The horizontal component F_h can be resolved further into two rectangular components F_x and F_z along the x and z axes respectively in the horizontal $OECD$, xz-plane (see Figure 7(c)). The components F_x and F_z are given by

$$F_x = F_h \cos\phi = |\boldsymbol{F}| \sin\theta_y \cos\phi \tag{10}$$

$$F_z = F_h \sin\phi = |\boldsymbol{F}| \sin\theta_y \sin\phi \tag{11}$$

The given force $\boldsymbol{F}$ has thus been resolved into three rectangular components of force F_x, F_y and F_z which are directed along the three coordinate axes.

Applying the Pythagorean theorem to the right angled triangles OAB and OCD, we get

$$F^2 = OA^2 = AB^2 + BO^2 = F_h^2 + F_y^2 \tag{12}$$

$$F_h^2 = OC^2 = CD^2 + OD^2 = F_z^2 + F_x^2 \tag{13}$$

Eliminating F_h^2 from the above two equations and solving for $\boldsymbol{F}$, we obtain the following relation between the magnitude of $\boldsymbol{F}$ and its components as follows:

$$|\boldsymbol{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{14}$$

The relationship existing between the force $\boldsymbol{F}$ and its three components F_x, F_y, and F_z is more easily visualized if a cuboid is drawn having the edges F_x, F_y, F_z as shown in the Figure 8. The force F is then represented by the diagonal of this cuboid.

Let, θ_x, θ_y, and θ_z denote the angles that the force F make with the three axes x, y and z respectively. We can thus write,

$$F_x = |\boldsymbol{F}|\, cos\, \theta_x, \quad F_y = |\boldsymbol{F}|\, sin\, \theta_y, \quad \text{and} \quad F_z = |\boldsymbol{F}|\, sin\, \theta_z \tag{15}$$

The three angles θ_x, θ_y, θ_z define the direction of the force F**;** they are more commonly used for this purpose than the angles θ_y and ϕ introduced at the beginning of this section. The cosines of θ_x, θ_y, θ_z are known as the direction cosines of the force F.

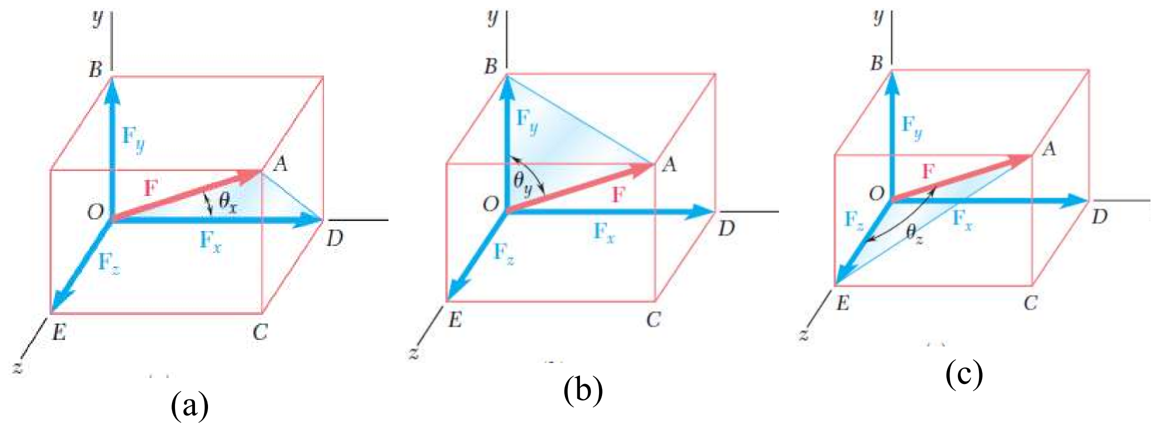


Figure 8: Cuboid containing the force ***F*** and its three components along the three edges and the angles made with respect to the three axes.

Introducing the unit vectors ***i***, ***j***, and ***k*** directed respectively along the *x, y* and *z* axes, we can express the force ***F*** in the following vector form:

$$\boldsymbol{F} = F_x\,\boldsymbol{i} + F_x\,\boldsymbol{j} + F_x\,\boldsymbol{k} \tag{16}$$

Concurrent Forces in Space: The resultant of two or more concurrent forces in space will be determined by finding out their rectangular components. Graphical or trigonometric methods are generally not practical in the case of forces in space. Using the principle of resolution of forces, we can write

$$R_x\,\boldsymbol{i} + R_y\,\boldsymbol{j} + R_z\,\boldsymbol{k} = \sum F_x\,\boldsymbol{i} + \sum F_y\,\boldsymbol{j} + \sum F_z\,\boldsymbol{k} \tag{17}$$

where, R_x, R_y, R_z denote the components of the resultant force along the three axes *x, y, z* and $\sum F_x$, $\sum F_y$, $\sum F_z$ represent the summation of components of the forces along the axes *x, y, z*. From the above equation (3), it follows,

$$R_x = \sum F_x \qquad R_y = \sum F_y \qquad R_z = \sum F_z \tag{18}$$

The magnitude of the resultant force *R* may be obtained as follows:

$$|\boldsymbol{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2} \tag{19}$$

The directions θ_x, θ_y, θ_z made by the resultant force *R* with the three axes *x*, *y*, *z* can be obtained as follows:

$$\cos\theta_x = \frac{R_x}{|\boldsymbol{R}|} \qquad \cos\theta_y = \frac{R_y}{|\boldsymbol{R}|} \qquad \cos\theta_z = \frac{R_z}{|\boldsymbol{R}|} \tag{20}$$

Graphical Method for the Equilibrium of Coplanar Forces: The equilibrium of such forces may be studied graphically by drawing vector diagram from the converse of the two laws we have already discussed:

1) Converse of the law/rule of triangle of forces, and 2) converse of the law/rule of polygon of forces.

For the case 1): Forces in a plane will be in equilibrium if three forces acting at a point be represented in magnitude and direction by the three sides of a triangle taken in order.

For the case 2): Forces in a plane will be in equilibrium if the number of forces acting at a point be represented in magnitude and direction by the sides of a closed polygon taken in order.

Types of Equilibrium: From the practical point of view, a body is said to be in equilibrium when it comes back to its original position after slightly displaced from its position of rest. There are following general three types of equilibrium:

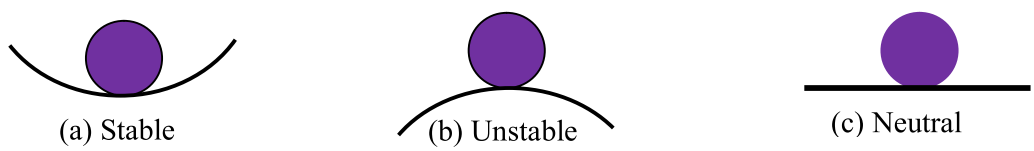


Figure 9: The three types of equilibrium: (a) stable, (b) unstable, (c) neutral.

1. *Stable equilibrium*: A body is said to be in stable equilibrium if it returns to its original position after it is slightly displaced from its position of rest. A smooth cylinder lying on a curved surface is an example of stable equilibrium which is depicted in Figure 9(a).

2. *Unstable equilibrium*: A body is said to be in an unstable equilibrium if it does not come to its original position and moves further away from its position of rest after the slight disturbance. Figure 9(b) shows an unstable equilibrium where a smooth cylinder lying on a convex surface. A slight disturbance to the cylinder will lead to move away from its original position.

3. *Neutral equilibrium*: A body is said to be in neutral equilibrium if it occupies a new position and remains at rest at this position after slightly displaced from its position of rest. For example, a smooth cylinder lying on a horizontal plane surface is in neutral equilibrium (see Figure 9(c)).

1. Numerical:

1. Two forces of 100 *N* and 150 *N* are acting simultaneously at a point. Find the magnitude and direction of the resultant force due to the two forces when the angle between them is 45°.

Solution: Given, let, F_1 = 100 *N*, F_2 = 150 *N*, and angle between the forces, θ = 45°.

We know from the Parallelogram law of forces,

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$= \sqrt{100^2 + 150^2 + 2 \times 100 \times 150 \times cos\,45^o}$$

$$= 232\ N$$

and direction of the resultant force, i.e. the angle that the resultant force *R* make with one of the force, let it is F_2.

$$tan\,\alpha = \frac{F_1\,sin\,\theta}{F_2 + F_1\,cos\,\theta}$$

$$tan\,\alpha = \frac{100 \times sin\,45^o}{150 + 100 \times cos\,45^o}$$

$$\alpha = tan^{-1}(0.32) \cong 18^o$$

Remark: You can also calculate the direction of the resultant force with respect to the force F_1 as follows (answer will be the same):

$$tan\,\alpha = \frac{F_2\,sin\,\theta}{F_1 + F_2\,cos\,\theta}$$

$$tan\,\alpha = \frac{150 \times sin\,45^o}{100 + 150 \times cos\,45^o}$$

$$\alpha = tan^{-1}(0.51) \cong 27^o$$

We know, 45° – 27° = 18°, which is the previous answer that we have obtained the direction of the resultant force *R* with respect to the force F_2.

2. A horizontal bar *PQRS* is 12 *m* long, where *PQ* = *QR* = *RS* = 4 *m*. Forces of 1000 *N*, 1500 *N*, 1000 *N* and 500 *N* respectively act at the points *P, Q, R, S* in the direction as shown in the Figure 1.1. The lines of action of these forces make angles of 90°, 60°, 45°, and 30° with the line *PS.* Find the magnitude, direction and position of the resultant force.

Solution:

1) Magnitude of the resultant force: Resolving the forces horizontally, we get

$$\sum H = 1000 \times \cos 90^o + 1500 \times \cos 60^o + 1000 \times \cos 45^o + 500 \times \cos 30^o$$

$$= 1890\ N$$

Now, resolving the forces vertically,

$$\sum V = 1000 \times \sin 90^o + 1500 \times \sin 60^o + 1000 \times \sin 45^o + 500 \times \sin 30^o = 3256\ N$$

The magnitude of the resultant force,

$$R = \sqrt{(\sum H)^2 + (\sum V)^2} = \sqrt{(1890)^2 + (3256)^2} = 3765\ N$$

2) Direction of the resultant force: Let, the resultant force *R* makes an angle α with the line *PS*.

$$\tan\alpha = \frac{\sum V}{\sum H} = \frac{3256}{1890} = 1.72$$

$$\alpha = tan^{-1}(1.72) = 59.8^{\circ}$$

3) Position of the resultant force: Let *x* be the distance between the point *P* the line of action of the resultant force *R*. Now, taking moments of the vertical component of the resultant force *R* and the forces about the point *P* and equating the same, we get

$$R \times \sin 59.8^{o} \times x = 1000 \times 0 + 1500 \times \sin 60^{o} \times 4 + 1000 \times \sin 45^{o} \times 8 + 500 \times \sin 30^{o} \times 12$$

$$3254 \times x = 13853$$

$$x = \frac{13853}{3254} = 4.25\ m$$

Note: Moment becomes zero when the line of action of a force passes through the point about which the moment is taken. In such scenario, the normal distance between the line of the force and the point becomes zero.