Chapter - 4 | Frequency Domain Analysis of Discrete Time Signals

Date - 9/06/21 Lecture Notes - Part 1

Note: Signals may be broadly classified into two types: Continuous-time signals and discrete-time signals. If we want to analyse both these two types of signals with the help of some mathematical tools, we will have to establish a separate analysis procedure for each of these two types of signals.

In this chapter, we shall analyze only ~~continuous~~ discrete time signals with the help of Fourier series and Fourier transform.

① When we analyze continuous time signals with the help of FS & FT, then FS & FT are called Continuous-time FS (CTFS) and Continuous time Fourier transform (CTFT).

② When we analyze ~~continuous~~ discrete time signals with the help of FS and FT, then FS & FT are called (DTFS) and (DTFT).

There are various types of transformation techniques - * mathematical tools

1) Laplace transform →
2) Fourier transform →

Common: Both are used to analyse signals & systems to convert time domain $x(t)$ to Freq domain $X(s)$ & $X(e^{j\omega})$.

3) Z transform → Z-domain

Advantage of LT over FT → applied to perform transform analysis on unstable system

Difference between time domain representation and frequency domain representation /

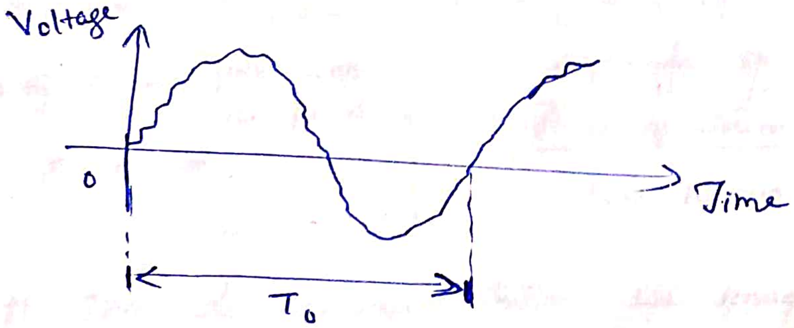
Frequency domain representation of a signal (Line spectrum)

Till now signals we have drawn were with respect to time.

⇒ CT time represented by → 't' | DT time represented by → 'n'

That means time 't' was considered as a variable.

* The representation of signal w.r.t time is called time domain representation.



$f_0 = 1/T_0$

Fig(a) Time domain representation

✓ This time domain representation of the signal is not sufficient for its analysis.

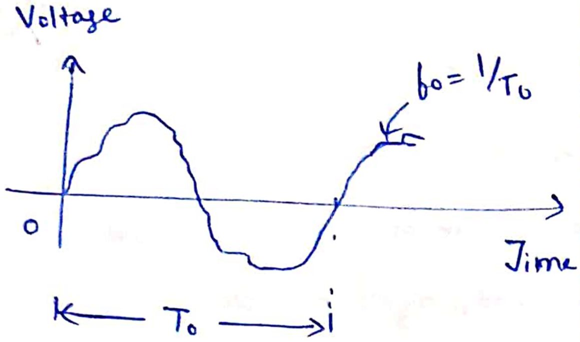
✓ So, for the sake of analysis we have to use frequency domain representation.

✓ In frequency domain representation, the variable plotted on X-axis is frequency 'f' rather than time 't'.

✓ So, the signal representated in the frequency domain is called line spectrum.

So, this transformation between time domain and frequency domain is performed using FT and DTFT.

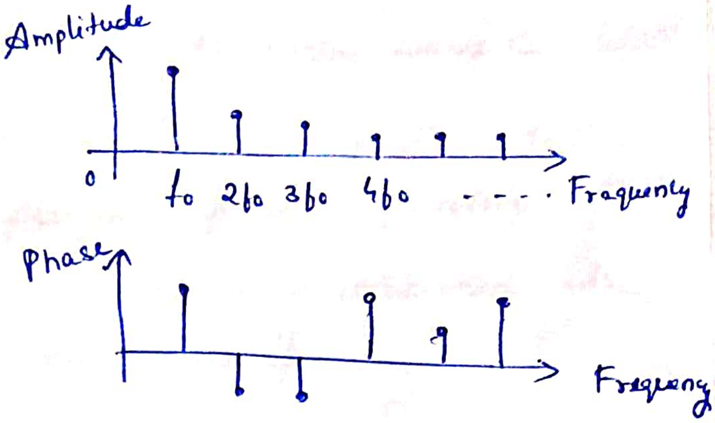
Time domain representation



This time domain representation gives us following information:

1) Shape of signal
2) Its frequency
3) Type of signal (periodic or non-periodic)
4) One cycle period

Frequency domain representation



Line spectrum consists of two graphs:

i) Amplitude spectrum: A graph of amplitude vs frequency.

ii) Phase spectrum: A graph of phase versus frequency.

✓ From the time domain representation we know nothing about frequency components and in what proportion they are mixed in order to get a particular shape of signal.

✓ All this information can be obtained from the line spectrum of a signal.

✓ So, after representation of signal in frequency domain using ~~either~~ Fourier Transform we obtain a line spectrum which consists of amplitude and phase of various frequency components present in the signal.

✓ Thus, the line spectrum allows us to analyse & synthesize a signal.

How to plot Line Spectrum ?

1) Independent variable plotted on x-axis is frequency 'f' in Hz.

2) Phase angle is always measured w.r.t cosine wave.

→ Necessary to convert sine wave to cosine wave –

$$\sin\omega t = \cos(\omega t - 90^\circ)$$

3) Amplitude is always regarded as positive quantity. So if negative sign appears, they should be absorbed in the phase change to keep the amplitude positive.

$$-A\cos\omega t = A\cos(\omega t \pm 180^\circ)$$

Here, additional phase change $\pm 180^\circ$ will convert the negative amplitude $-A$ to positive amplitude $+A$.

→ We can choose either $+180^\circ$ or -180° as the effect is going to be same.

Ex 1: Sketch the line spectrum of following signal:

$$m(t) = 3 - 5\cos(40\pi t - 30^\circ) + 4\sin 120\pi t$$

Sol:) In the given signal, first term is a dc term which has zero frequency.

1) First term: $3 = 3\cos 2\pi 0 t$, as $f = 0$

2) Second term: $-5\cos 40\pi t - 30^\circ = 5\cos(2\pi 20t - 30^\circ + 180^\circ)$

$$= 5\cos(2\pi 20t + 150^\circ)$$

3) Third term: $4\sin 120\pi t = 4\sin 2\pi 60t = 4\cos(2\pi 60t - 90^\circ)$

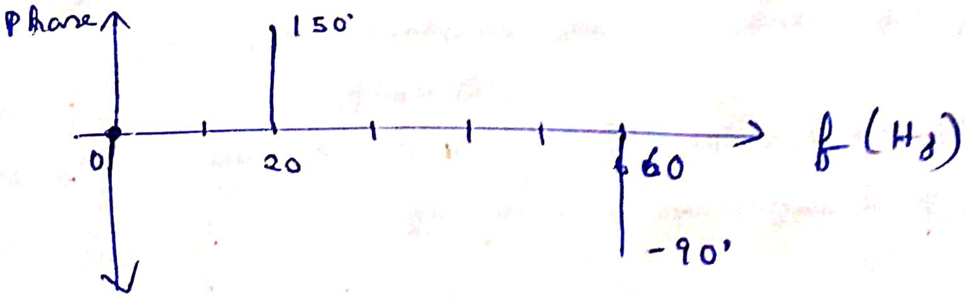
In ② & ③ term, $+180^\circ$ phase angle is added to make the amplitude positive & -90° phase shift is added to convert sine term to cosine term.

S.No.	Term	Amplitude	Frequency	Phase
1.	$3\cos 2\pi 0 t$	3V	0 Hz	0°
2.	$5\cos(2\pi\, 20t + 150)$	5V	20 Hz	150°
3.	$4\cos(2\pi\, 60t - 90)$	4V	60 Hz	−90°

Plot the line spectrum: (Single sided spectrum)
one sided "
positive frequency line spectra.

(a) **Amplitude spectrum:** Amplitude vs frequency

(b) **Phase spectrum:** Phase vs frequency

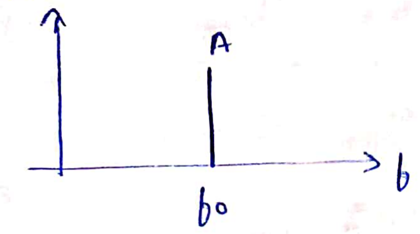


Ex:

Double - sided spectrum:

→ It consists of negative frequencies along with positive frequencies

Single sided



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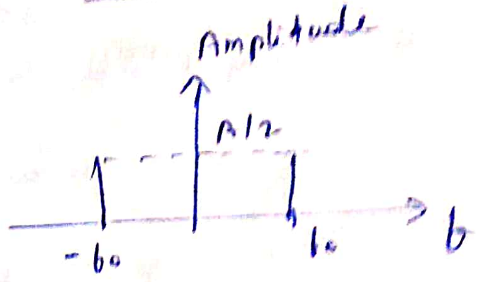
Double sided

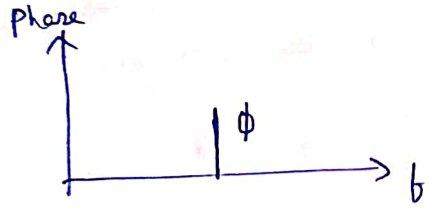


i) In amplitude vs frequency graph → amplitude spectrum

{ Single Sided → there is only one frequency component $f = f_0$ with amplitude A

⇒

{ double sided → there is two frequency components f_0 & $-f_0$



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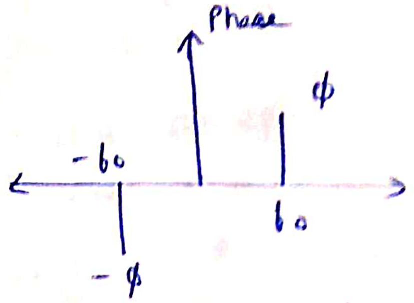


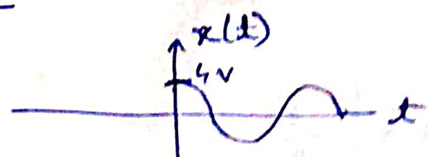
ii) In phase spectrum:

Single sided → contains one component at f_0 with phase ϕ_0

double sided → contains two components at f_0 & $-f_0$ with phases equal to ϕ & $-\phi$.

<u>Ex:</u> $x(t) = 10 \cos 200\pi t$

1) Time domain →



2) Frequency domain → $10 \times \left[\frac{e^{2\pi f j t} + e^{-2\pi f j t}}{2} \right]$

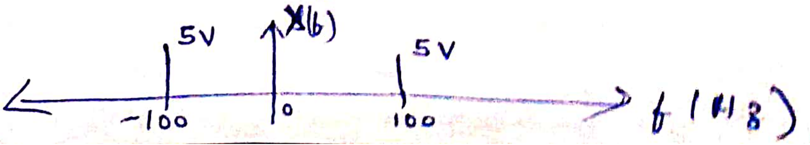
$X(f) = 5 \left[e^{j2\times 3.14\,28 \times 100t} + e^{-j2\times 3.1428\times 100t} \right]$

$= 5 \left[e^{j628.28t} + e^{-j628.28t} \right]$

Hint:
$2\pi f = 200\pi$
$f = 100$ Hz

~~$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{}$~~

Using:
Euler's theorem.



Fourier Series: (CTFS) → [Not in syllabus]
(Only DTFS required)

Note: Any periodic signal basically consists of sine waves having different amplitudes of different frequencies and having different relative phase shifts.

Trigonometric fourier series: (or) (exponential)

Fourier series represents a periodic waveform in the sum of infinite number of sine and cosine terms.
It is a representation of signal in time domain series form.
✓ Fourier series is a mathematical tool used to analyze any periodic signal.

→ A periodic function $x(t)$ can be expressed in form trigonometric fourier series comprising the sine and consine terms.

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_n \cos n\omega_0 t + \dots + b_1 \sin\omega_0 t + b_2 \sin\omega_0 t + \dots b_n \sin n\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

for a period (t_0, t_0+T) → function repeating after a particular period T. same $(-\infty, \infty)$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{T}\int x(t)\,dt$$

If $x(t)$ is even (symmetric) then only cosine terms are present

$$\to a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt.$$

↓
if odd $x(t)$ only sine terms are present.

1) Exponential form of Discrete Fourier Series of a Periodic Sequence.

A real periodic discrete-time signal $x(n)$ of period N can be expressed as a weighted sum of exponential sequences.

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} = \sum_{k=0}^{N-1} C_k e^{j\omega_0 n}.$$

for $k = 0, 1, 2, \ldots n-1$

$-\infty < n < \infty$

where the term C_k is fourier coefficient & is given by

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \frac{1}{N} x(n) e^{-j\omega_0 n}$$

Here; $\omega_0 = \frac{2\pi k}{N}$

→ Each of fourier series coefficient C_k represents the amplitudes and phase of phasor components.

→ Fourier series represents the spectrum of the signal

∴ For a frequency k/N, fourier series representation is the frequency domain representation of signal $x(n)$.

Example: Evaluate the fourier series of signal

$$x(n) = \{ \underset{\uparrow}{1}, 1, 0, 0 \}$$

Periodicity $N = 4$

Sol:

$$x(n) = \sum_{k=0}^{N-1} c_k \, e^{j2\pi kn/N}$$

$k = 0$ to $N-1 = 4-1 = 3$

$$x(n) = \sum_{k=0}^{3} c_k \, e^{j2\pi kn/4}$$

Since $N = 4$,

Fourier coefficients,

$$c_k = \frac{1}{4} \sum_{n=0}^{3} x(n) \, e^{-j2\pi kn/4}$$

$$= \frac{1}{4} \sum_{n=0}^{3} x(n) \, e^{-j\pi kn/2}$$

For all samples of $x(n)$, $n = 0, 1, 2, 3$

$x(0) = 1$, $x(1) = 1$, $x(2) = 0$, $x(3) = 0$.

$$c_k = \frac{1}{4} \left\{ x(0) \, e^{-j\pi k \cdot \frac{0}{2}} + x(1) \, e^{-j\pi k \frac{1}{2}} + 0 + 0 \right\},$$

$$= \frac{1}{4} \left\{ 1 + 1 \cdot e^{-j\pi k/2} \right\}$$

$$= \frac{1}{4} \left\{ 1 + e^{-j\pi k/2} \right\}.$$

For $k = 0, 1, 2, 3$.

Substituting the values of k, ~~k=0~~

For $k = 0$

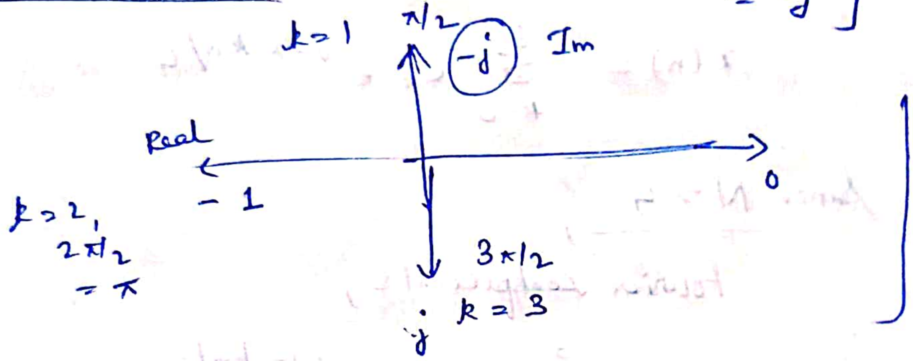
$$C_0 = \frac{1}{4}\{1 + e^{0}\} = \frac{1}{4}\{1+1\} = \frac{2}{4} = \frac{1}{2}$$

For $k = 1$

$$C_1 = \frac{1}{4}\left[1 + e^{-j\pi/2}\right] = \frac{1}{4}[1 - j]$$

Hint:

Since at $k = 1$, $e^{-j\pi/2}$ is imaginary value $= -j$



Hint

$$e^{j\pi/2} = \cos\pi/2 - j\sin\pi/2 = 0 - j = -j$$

For $k = 2$, $C_2 = 0$, since at $k = 2$, $e^{-j2\pi/2} = e^{-j\pi}$ is real value $= -1$

For $k = 3$, $C_3 = \frac{1}{4}(1 + j)$, since at $k = 3$ $e^{-j3\pi/2}$ is imaginary value $= j$

Evaluating the magnitude of F.S coeff, we get

	$\lvert C_0\rvert = \frac{1}{2}$	$\lvert C_1\rvert = \frac{\sqrt{2}}{4}$	$\lvert C_2\rvert = 0$	$C_3 = \frac{\sqrt{2}}{4}$
Respective frequencies,	$f_0 = 0$	$f_1 = \frac{1}{4}$	$f_2 = 2/4$	$f_3 = 3/4$
$(\omega = 2\pi f)$ relation	$(\because \omega_0 = 0)$	$(\because \omega_1 = \pi/2)$	$(\because \omega = \pi)$	$(\because \omega_3 = 3\pi/2)$

Sketch the spectrum