Chapter 2

Lecture Plan (26th May 2021)

1. Remaining properties of LTI systems:

Short notes (Previous years paper 2016-19)

- *Causality for LTI systems*
- *Stability for LTI systems*
- *Stability and Causality of LTI system described by differential/difference equations*

2. Correlation function (2016,17)

Question- *How* ***Radar*** *system works on basis of correlation function or*
Question - Explain any physical process that requires the signal processing techniques of cross correlation

Two more properties of LTI System-

1. Causality

2. Stability

(Charecterize in terms of impulse response h(n))

Explanation:

1. Conditon of Causality of LTI System

Hint- Speaker notes

Word causality has come from word cause

'Every event must have a cause in past time'

- In the case of an LTI system, one can determine whether or not it is causal by looking at its unit impulse response.
- **Fact 1 (for CT systems):** A CT LTI system is causal if and only if its unit impulse response h(t) satisfies

$$h(t)=0 \text{ for } t<0.$$

- **Fact 1 (for DT systems):** A DT LTI system is causal if and only if its unit impulse response h[n] satisfies

  $h[n]=0$ for $n<0$.

In the case of LTI system, causality can be translated to a condition on the impulse response.

For a Discrete time LTI system a Convolution Sum is given by-

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{——— (1)}$$

(or) $y(n) = x(n) * h(n)$

$y(n) = h(n) * x(n)$

Using Commutative property,

we rewrite the eqn (1) as

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{--- (2)}$$

On expanding, the summation limits will be $k = -\infty$ to -1

$k = 0$ to ∞

Eqn (2) becomes

$$y(n) = \sum_{k=-\infty}^{-1} h(k)\, x(n-k) + \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

$$y(n) = \underbrace{\ldots + h(-2)\,x(n+2) + h(-1)\,x(n+1)}_{\text{(I) future i/p's}}$$

$$\underbrace{+ h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots}_{\text{(II) Present \& past i/p's}}$$

✓

Hence for a causal system, the impulse response must satisfy the following condition,

$$h(n) = 0, \quad n < 0$$

So, the limits of summation of the convolution formula will be modified to $k = 0$ to ∞,

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

for $h(n) = 0, \; n < 0$

$$\text{(or)} \quad y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k)$$

2. Conditon of Stability of LTI System

Recall-

Definition of a (BIBO) "stable" sytem.

'A "stable system" is a system whose response is bounded for any bounded input'.

What is the Condition ?

In discrete time LTI system is stable if its impulse response is always summable.

Stability for LTI system-

$$\sum_{n=-\infty}^{\infty} | h[n] | < \infty$$

If input, x(n) is bounded, there exists a constant (positive integer) M_x such that

$$|x(n) \leq M_x < \infty \quad \text{——— (1)}$$

Similarly, if output, $y(n)$ is bounded there exist a constant (+ve) M_y such that

$$|y(n)| < M_y < \infty \quad \text{——} (2)$$

Convolution sum formula, is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{——} (3)$$

If we take the absolute value of both sides of eq^n, we obtain

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right| \quad \text{——} (4)$$

Now the absolute value of the sum of terms is always less than or equal to the sum of absolute value of the terms.

Hence, (By triagular inequality)

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \quad \text{——— (5)}$$

From eqn (1), if the input is bounded there exists a finite number M_x such that

$|x(n)| \leq M_x$.

By substituting this upper bound for $x(n)$ in eqn (5), we obtain

$$|y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)| \quad \text{——— (6)}$$

From this expression, we observe that output is bounded if the impulse response of the system satisfies the condition

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad (7)$$

∴ A linear time invariant system is stable if its impulse response is absolutely summable. //

3. Stability and Causality described by Difference Equations

(a) *Linear Constant-Coefficient Difference Equations*

In a causal LTI difference system, the discrete-time input and output signals are related implicitly through a linear constant-coefficient difference equation.

In general, an Nth-order linear constant coefficient difference equation has the form-

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{——— (1)}$$

The solution of the differential equation can be obtained when we have the N initial conditions (or auxiliary conditions) on the output variable.

The solution to the difference equation is the sum of the homogeneous solution

$$\sum_{k=0}^{N} a_k y[n-k] = 0 \qquad (2)$$

(a solution with input set to zero, or natural response) and of a particular solution (a function that satisfy the difference equation).

The concept ofinitial rest of the LTI causal system described by difference equation means that

$x[n] = 0 , n < n_0$

implies $y[n] = 0 , n < n_0$

In fact, if $N \geq 1$ in Eq. (1), the difference equation is recursive, it is usually the case that the LTI system corresponding to this equation together with the condition of initial rest will have an impulse response of infinite duration.

Such systems are referred to as infinite impulse response (IIR) system.

(b) Block Diagram Representations of 1st -order Systems Described by Differential and Difference Equations

Block diagram interconnection is very simple and nature way to represent the systems described by linear constant-coefficient difference and differential equations.

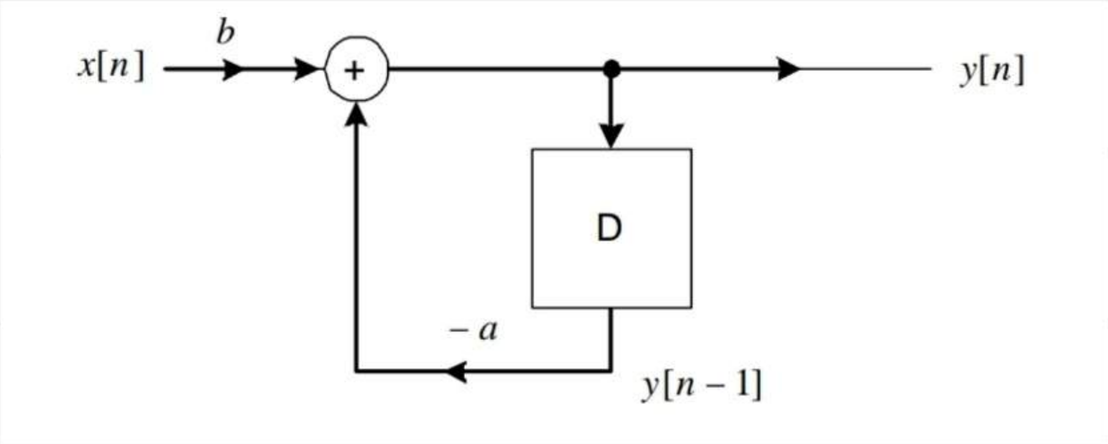
For example, the causal system described by the first-order difference equation is

$$y[n] + ay[n-1] = bx[n].$$

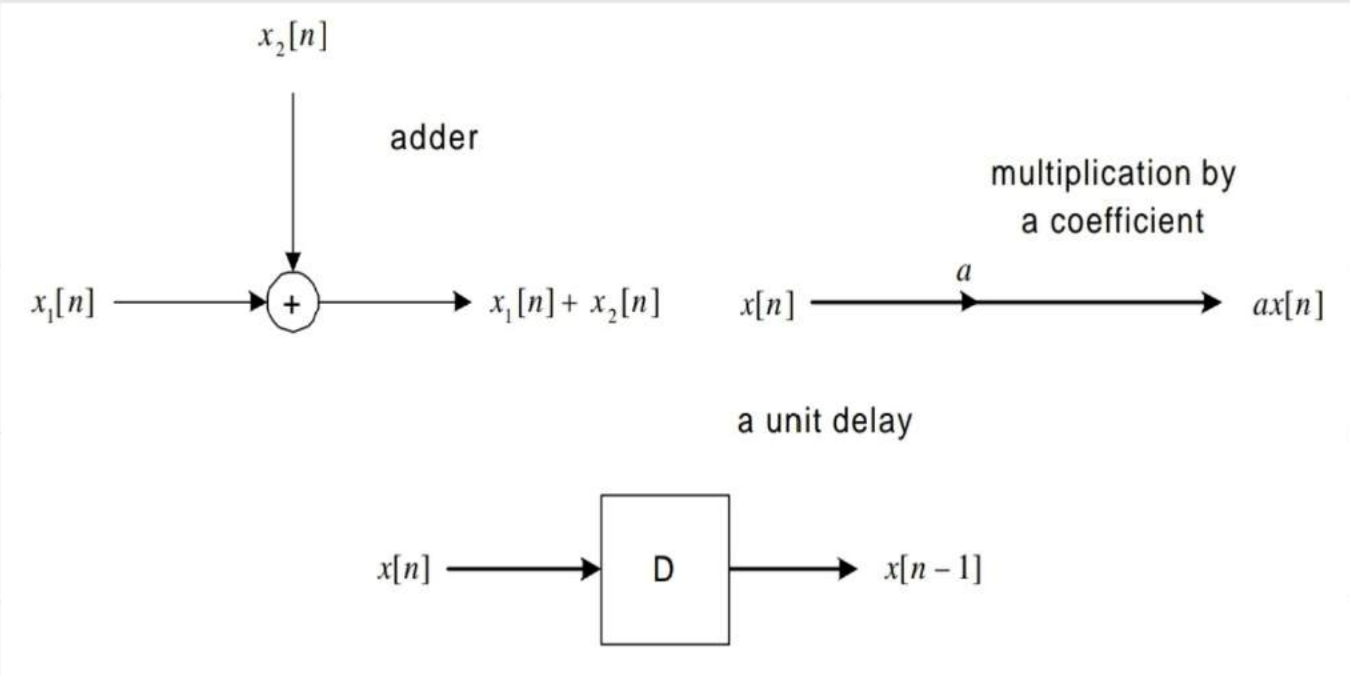
It can be rewritten as

$$y[n] = -ay[n-1] + bx[n]$$

The block diagram representation for this discrete-time system is show



Three elementary operations are required in the block diagram representation: addition, multiplication by a coefficient, and delay.



Question- *How* ***Radar*** *system works on basis of correlation function*

Or

Question - Explain any physical process that requires the signal processing techniques of cross correlation

Answer - The concept of correlation can best be presented with an example.
Figure below shows the key elements of a radar system. A specially designed antenna transmits a short burst of radio wave energy in a selected direction. If the propagating wave strikes an object, such as the helicopter in this illustration, a small fraction of the energy is reflected back toward a radio receiver located near the transmitter. The transmitted pulse is a specific shape that we have selected, such as the triangle shown in this example. The received signal will consist of two parts: (1) a shifted and scaled version of the transmitted pulse, and (2) random noise, resulting from interfering radio

waves, thermal noise in the electronics, etc. Since radio signals travel at a known rate, the speed of light, the shift between the transmitted and received pulse is a direct measure of the distance to the object being detected. This is the problem: given a signal of some known shape, what is the best way to determine where (or if) the signal occurs in *another* signal. Correlation is the answer.

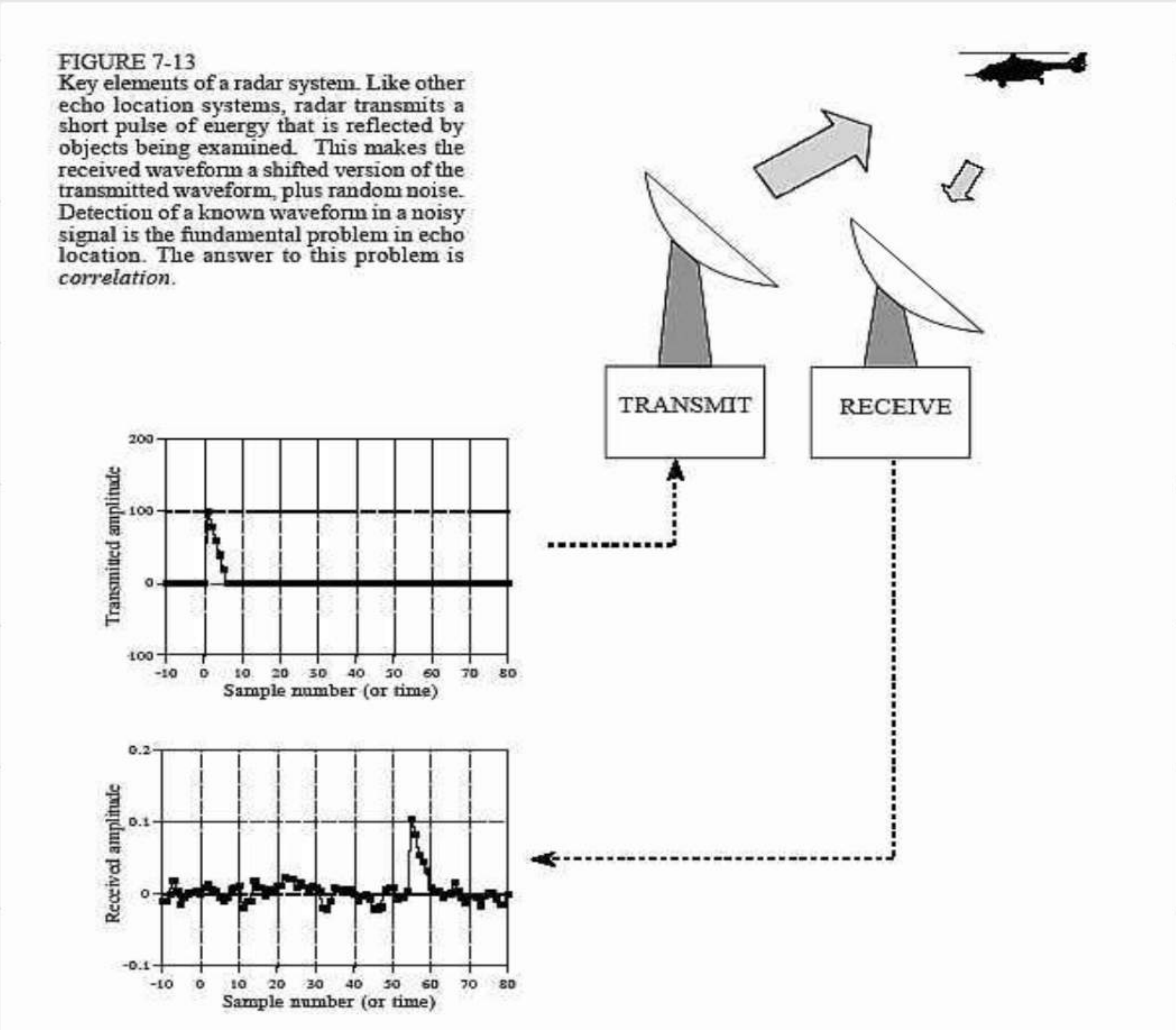


FIGURE 7-13
Key elements of a radar system. Like other echo location systems, radar transmits a short pulse of energy that is reflected by objects being examined. This makes the received waveform a shifted version of the transmitted waveform, plus random noise. Detection of a known waveform in a noisy signal is the fundamental problem in echo location. The answer to this problem is *correlation*.

Correlation is the *optimal* technique for detecting a known waveform in random noise. That is, the peak is higher above the noise using correlation than can be produced by any other linear system. (To be perfectly correct, it is only optimal for *random white noise*). Using correlation to detect a known waveform is frequently called matched filtering.

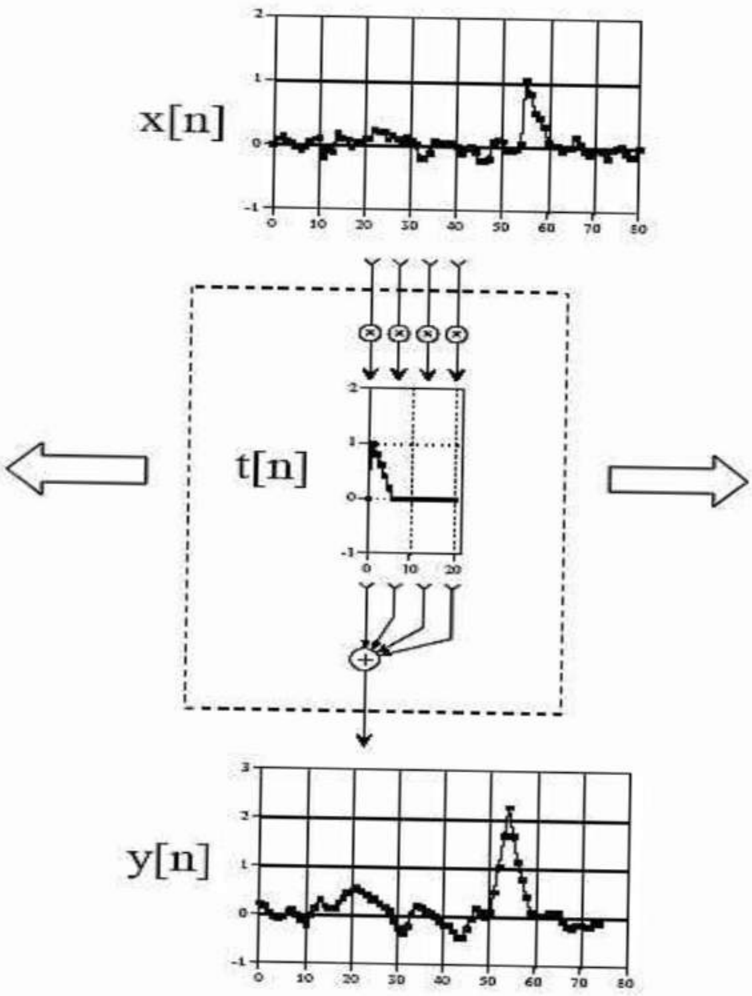


FIGURE 7-14
The correlation machine. This is a flowchart showing how the cross-correlation of two signals is calculated. In this example, $y[n]$ is the cross-correlation of $x[n]$ and $t[n]$. The dashed box is moved left or right so that its output points at the sample being calculated in $y[n]$. The indicated samples from $x[n]$ are multiplied by the corresponding samples in $t[n]$, and the products added. The correlation machine is identical to the convolution machine (Figs. 6-8 and 6-9), except that the signal inside of the dashed box is *not* reversed. In this illustration, the only samples calculated in $y[n]$ are where $t[n]$ is fully *immersed* in $x[n]$.

Alternative Answer

Objectives:

(a) Understand the basic theory behind RADAR systems.

(b) Understand and implement the cross-correlation for a simple Radar system.

(c) Measure the time delay by computing cross-correlation.

(d) Calculate the distance from the radar to the target.

Introduction:

Cross-correlation is a measure of similarity of two signals as a function of time delay applied to one of them. Auto-correlation is a cross-correlation between the signal and itself. The correlation functions are used in many applications in communication systems. For example, in Radar and Sonar systems, it can be used to determine the delay between the transmitted and received signals; consequently, the distance between the target and the Sonar / Radar can be determined as well.

Radar systems:

The physical principle on which radar operates is very similar to the principle of sound-wave reflection. If you shout in the direction of a sound-reflecting object (like a rocky canyon or cave), you will hear an echo. If you know the speed of sound in air, you can then estimate the distance and general direction of the object. The time required for an echo to return can be roughly converted to distance if the speed of sound is known.

Radar uses electromagnetic energy pulses in much the same way, as shown in Figure 1 and 2. The radio-frequency (RF) energy is transmitted to and reflected from the reflecting object. A small portion of the reflected energy returns to the radar set. This returned energy is called an ECHO, just as it is in sound terminology. Radar sets use the echo to determine the direction and distance of the reflecting object.

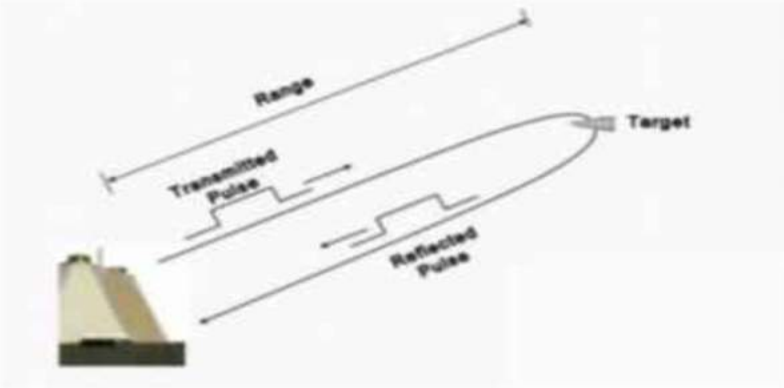


Figure 1: radar system

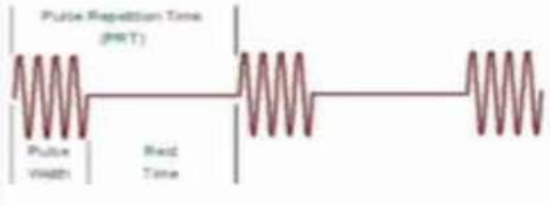


Figure 2: radio pulse signal

Let $x(t)$ be the transmitted signal and $y(t)$ the received signal of a radar system; the two signals are related as follows:

$$y(t) = ax(t - d) + n(t) \quad (1)$$

where a is the attenuation, d is the delay and $n(t)$ is an Additive White Gaussian Noise term of zero mean and variance σ^2. The signals $x(t)$ and $y(t)$ are sampled at the receiver and are processed digitally to determine the time delay and hence the distance between the radar and the object. If we sample the received signal with a sampling time T, then the resulting discrete-time signal is:

$$y(nT) = ax(nT - DT) + n(nT) \quad (2)$$

Hence, by omitting the sampling time T, we obtain the discrete time sequence:

$$y[n] = ax[n-D]+n[n] \quad (3)$$

If we calculate the auto-correlation between both sequences y[n] and x[n], that is:

$$r_{yx}[k] = y[n] \oplus x[n] \quad (4)$$

then, the maximum of the cross-correlation function indicates the time delay where the signals are best aligned.

$$D = \text{argmax}_k(r_{yx}[k]) \quad (5)$$

Where "argmax" is the argument of the maximum cross-correlation.

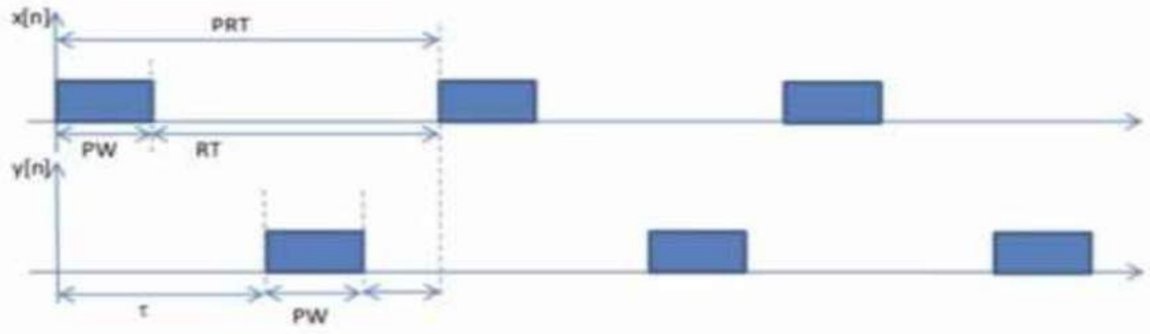
Then, the discrete time delay D is converted to continuous time delay as follows:

$$d = D.T \quad (6)$$

Finally, the distance R is obtained by using the following equation:

$$R = \frac{cd}{2} \quad (7)$$

where c is the speed of light and d is the delay.



Description of Tasks

Over one pulse repetition time PRT = 8, do the following tasks:

1. **Task1:** plot $x[n]$ where the pulse sequence is a Barker sequence given by: {+1 +1 −1}. Note that the rest time (RT) is 5.
2. **Task2:** consider $\tau = 2$ and $a = 0.9$. Plot the resultant sequence $y[n]$.
3. **Task3**: Compute and plot the cross-correlation $r_{yx}(k)$ and compute the delay D using Eq. (5).
4. **Task4**: determine d and R using equations (6) and (7) above if T = 10μS.
5. **Task5**: (bonus): add additive white Gaussian noise (AWGN) to $y[n]$ in task 2 above and perform task 3 and 4; consider three scenarios: a) AWGN with zero mean and variance $\sigma = 0.1$ b) AWGN with zero mean and variance $\sigma = 0.9$ and c) AWGN with zero mean and variance $\sigma = 5$.

Compare and comment.

2.15 DISCRETE CORRELATION

Till now we have discussed about the convolution of two signals. It is used to find the output sequence $y(n)$ if the input sequence $x(n)$ and the impulse response $h(n)$ are known. Here we discuss about correlation which is a mathematical operation similar to convolution. Correlation is used to compare two signals. It is a measure of similarity between signals and is found using a process similar to convolution. It occupies a significant role in signal processing. It is used in radar and sonar systems to find the location of a target by comparing the transmitted and reflected signals. Other applications of correlation are in image processing, control engineering etc. The correlation is of two types: (i) Cross correlation (ii) Auto-correlation.

2.15.1 Cross Correlation

The cross correlation between a pair of sequences $x(n)$ and $y(n)$ is given by

$$R_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)\,y(k-n)$$

$$= \sum_{k=-\infty}^{\infty} x(k+n)\,y(k) \qquad n = 0, \pm 1, \pm 2, \ldots$$

The index n is the shift (lag) parameter. The order of subscripts xy indicates that $x(n)$ is the reference sequence that remains fixed, i.e. unshifted in time, whereas the sequence $y(n)$ is shifted n units in time w.r.t. $x(n)$.

If we wish to fix the sequence $y(n)$ and to shift the sequence $x(n)$, then the correlation can be written as:

$$R_{yx}(n) = \sum_{k=-\infty}^{\infty} y(k)\,x(k-n)$$

$$= \sum_{k=-\infty}^{\infty} y(k+n)\,x(k) \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$R_{xy}(n) \neq R_{yx}(n)$$

If the time shift $n = 0$, then we get

$$R_{xy}(0) = R_{yx}(0) = \sum_{k=-\infty}^{\infty} x(k)\,y(k)$$

Comparing the above equations for $R_{xy}(n)$ and $R_{yx}(n)$, we observe that

$$R_{xy}(n) = R_{yx}(-n)$$

where $R_{yx}(-n)$ is the folded version of $R_{yx}(n)$ about $n = 0$.

The expression for $R_{xy}(n)$ can be written as:

$$R_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)\,y[-(n-k)]$$

$$= x(n) * y(-n)$$

Observing the above equation for $R_{xy}(n)$, we can conclude that the correlation of two sequences is essentially the convolution of two sequences in which one of the sequence has been reversed, i.e., correlation is the convolution of one signal with a flipped version of the other. Therefore, the same algorithm (procedure) can be used to compute the convolution and correlation of two sequences.

2.15.3 Computation of Correlation

The correlation of two sequences $x(n)$ and $y(n)$ can be obtained by using the procedures for computing the convolution.

We have $$R_{xy}(n) = x(n) * y(-n)$$

So to get the cross correlation $R_{xy}(n)$, first fold the sequence $y(n)$ to obtain $y(-n)$. Then the convolution of $x(n)$ and $y(-n)$ gives the value $R_{xy}(n)$.

Also we have $$R_{xx}(n) = x(n) * x(-n)$$

So to get the autocorrelation $R_{xx}(n)$, first fold the sequence $x(n)$ to obtain $x(-n)$. Then the convolution of $x(n)$ and $x(-n)$ gives the value $R_{xx}(n)$.

EXAMPLE 2.46 Find the cross correlation of two finite length sequences:

$$x(n) = \{2, 3, 1, 4\} \quad \text{and} \quad y(n) = \{1, 3, 2, 1\}$$

Solution: Given $x(n) = \{2, 3, 1, 4\}$ and $y(n) = \{1, 3, 2, 1\}$

$$x(n) = \{2, 3, 1, 4\},\ y(n) = \{1, 3, 2, 1\},\ y(-n) = \{1, 2, 3, 1\}$$

$$R_{xy}(n) = x(n) * y(-n)$$

The cross correlation is computed as given below.

$x(n)$ \ $y(-n)$	1	2	3	1
2	2	4	6	2
3	3	6	9	3
1	1	2	3	1
4	4	8	12	4

$$R_{xy}(n) = \{2, 3+4, 1+6+6, 4+2+9+2, 8+3+3, 12+1, 4\} = \{2, 7, 13, 17, 14, 13, 4\}$$

EXAMPLE 2.48 Find the cross correlation of the sequences

$$x(n) = \{3, \underset{\uparrow}{5}, 1, 2\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{1}, 4, 3\}$$

and show that $R_{xh}(n) \neq R_{hx}(n)$ and $R_{xh}(n) = R_{hx}(-n)$.

Solution: *Computation of* $R_{xh}(n)$

We know that $R_{xh}(n) = x(n) * h(-n)$. So we compute the convolution of $x(n)$ and $h(-n)$.

Here $x(n) = \{3, \underset{\uparrow}{5}, 1, 2\}$ and $h(-n) = \{3, 4, \underset{\uparrow}{1}\}$. So the starting index of $R_{xh}(n)$ is $n = -1 - 2 = -3$. $R_{xh}(n)$ is computed using the sum-by-column method for convolution as follows:

n	-3	-2	-1	0	1	2
$x(n)$	3	5	1	2		
$h(-n)$	3	4	1			
	9	15	3	6		
		12	20	4	8	
			3	5	1	2
$R_{xh}(n)$	9	27	26	15	9	2

So $R_{xh}(n) = \{9, 27, 26, 15, 9, 2\}$

2.15.2 Autocorrelation

The autocorrelation of a sequence is correlation of a sequence with itself. It gives a measure of similarity between a sequence and its shifted version. The autocorrelation of a sequence $x(n)$ is defined as:

$$R_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)x(k-n)$$

or equivalently

$$R_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k+n)x(k)$$

If the time shift $n = 0$, then we have

$$R_{xx}(0) = \sum_{k=-\infty}^{\infty} x^2(k)$$

$R_{xx}(n)$ is given by $R_{xx}(n) = x(k) * x(-k)$

The autocorrelation has a maximum value at $n = 0$ and satisfies the inequality

$$|R_{xx}(n)| \leq R_{xx}(0) = E_x$$

The autocorrelation is an even symmetric function with

$$R_{xx}(n) = R_{xx}(-n)$$

Correlation is an effective method of detecting signals buried in noise. Noise is essentially uncorrelated with the signal. This means that if we correlate a noisy signal with itself, the correlation will be due only to the signal (if present) and will exhibit a sharp peak at $n = 0$.

EXAMPLE 2.47 Find the autocorrelation of the finite length sequence $x(n) = \{2, 3, 1, 4\}$.

Solution: Given

$$x(n) = \{2, 3, 1, 4\}$$

$$x(n) = \{2, 3, 1, 4\}, \; x(-n) = \{4, 1, 3, 2\}$$

$$R_{xx}(n) = x(n) * x(-n)$$

The autocorrelation is computed as given below:

$x(n)$ \ $x(-n)$	4	1	3	2
2	8	2	6	4
3	12	3	9	6
1	4	1	3	2
4	16	4	12	8

$$R_{xx}(n) = \{8, 12 + 2, 4 + 3 + 6, 16 + 1 + 9 + 4, 4 + 3 + 6, 12 + 2, 8\}$$

$$= \{8, 14, 13, 30, 13, 14, 8\}$$