RELIABILITY FUNCTION DERIVATION

Reliability is defined as the probability that a system (component) will function over some period of time ‘t’. To express this relationship mathematically, we define the continous randon variable ‘T’ to be the time to failure of the system (or component), T ≥ 0.

Then, reliability can be expressed as

$$R(t) = \text{Prob}\{T \geq t\}$$

Where $R(T) \geq R(0) = 1 \text{ and } \lim_{t \to \infty} R(t) = 0$ ... (1)

For a given value of t, $R(t)$ is the probability that the time to failure is greater than or equal to ‘t’.

We define,

$$F(t) = 1 - R(t) = \text{Prob}\{T < t\}$$

Where $F(0) = 0$,

and $\lim_{t \to \infty} F(t) = 1$ ... (2)

then, F (t) is the probability that a failure occurs before time ‘t’. Thus, R (t) is the reliability function and F (t) is the cumulative distribution function (CDF) of the failure distribution. A third function called probability density function (PDF) is defined as

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \quad \text{... (3)}$$

This function describes the shape of the failure distribution. The three functions are illustrated in the Fig. below

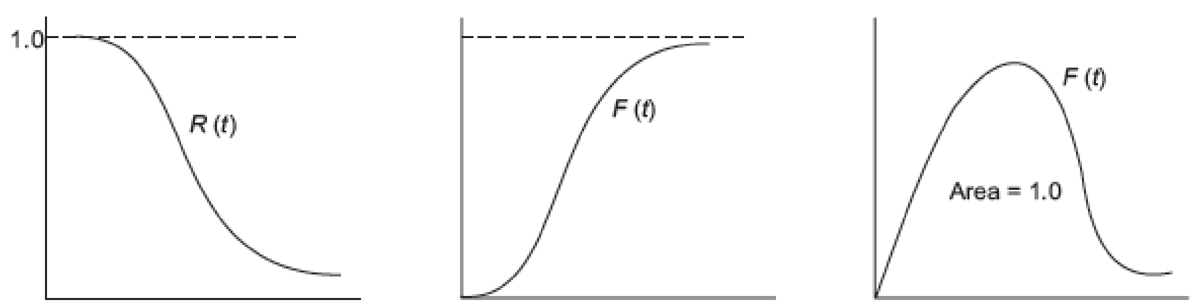

Fig: (a) Reliability function (b) Cumulative distribution function (c) Prob. density function.

The probability density function (*pdf*), has the following two properties

$$f(t) \geq 0 \text{ and } \int_0^{\infty} f(t)dt = 1$$

Given *Pdf* . $f(t)$, then

$$F(t) = \int_0^t f(t')dt'$$

$$R(t) = \int_t^{\infty} f(t')dt'$$

Both reliability function R (t), and the cumulative density function represent areas under the curve defined by F (t). Since the area under the curve is equal to one, both the reliability and failure probability will be defined so that,

O ≤ R(t) ≤ 1 and O ≤ F(t) ≤ 1

The function R (t) is normally used when reliabilities are being computed and the function F (t) is normally used when the failure probabilities are being computed. The graphical representation of pdf [f (t)] provides a visual representation of the failure distribution.

Relation Between $f(t)$, $R(t)$ and $\lambda(t)$

1. Probability Density Function [$f(t)$]

It is the probability that a random trial yields the value of 't' within the interval from t_1 to t_2 and expressed as

$$\int_{t_1}^{t_2} f(t)\, dt$$

$f(t)$ is the density function for a continuous random variable.

2. Distribution Function [$F(t)$]

It is the probability that in a random trial, the random variable is not greater than 't'.

$$\therefore \qquad F(t) = \int_{-\infty}^{t} f(t)\, dt$$

$F(t)$ is recognized as unreliability function.

3. Reliability [$R(t)$]

It expresses the probability that the variable is at least as large as 't'.

$$R(t) = \int_{t}^{\infty} f(t)\, dt$$

and

$$R(t) = 1 - F(t).$$

4. Failure Rate [λ. (*t*)]

The rate at which failures occur in the interval t_1 and t_2 is called the failure rate. It is expressed as the conditional probability that failures occur in the interval t_1 and t_2, given that failures have not occurred prior to t_1, *i.e.*, the start of the interval.

Failure rate is given by

$$\lambda(t) = \frac{\int_{t_1}^{t_2} f(t)\,dt}{(t_2 - t_1)\int_{t_1}^{\infty} f(t)\,dt}$$

$$= \frac{\int_{t_1}^{\infty} f(t)\,dt - \int_{t_2}^{\infty} f(t)\,dt}{(t_2 - t_1)\int_{t_1}^{\infty} f(t)\,dt}$$

$$= \frac{R(t_1) - R(t_2)}{R(t_1)(t_2 - t_1)}$$

If we substitute $t_1 = t$ and $t_2 = t + \Delta t$

The failure rate can be expressed as

$$\lambda(t) = \frac{R(t) - R(t + \Delta t)}{\Delta t\, R(t)}$$

As a special case,

where $f(t)$, the probability density function is exponential the various functions can be computed as

Probability density function $f(t)$ is

$$f(t) = (\lambda e^{-\lambda t})$$

where λ is the constant failure rate.

$$f(t) = \int_0^t \lambda e^{-\lambda t}\,dt$$

$$= 1 - e^{-\lambda t}$$

Reliability function can be calculated as

$$R(t) = \int_t^{\infty} \lambda e^{-\lambda t}\,dt$$

$$= e^{-\lambda t}$$

$$R(t) = 1 - F(t)$$

HAZARD RATE Z (t)

Hazard rate or instantaneous failure rate is defined as the limit of failure rate as the time interval length approaches to zero. It is a measure of instantaneous speed of failure.

It is expressed as

$$Z(t) = \lim_{\Delta t \to 0}\left[\frac{R(t) - R(t+\Delta t)}{\Delta t\, R(t)}\right]$$

$$= \frac{-1}{R(t)}\frac{d(Rt)}{dt}$$

$$\therefore \quad Z(t) = \frac{f(t)}{R(t)}$$

as

$$f(t) = \frac{-d\,R(t)}{dt}$$

Relationship between $R(t)$ and $Z(t)$

$$Z(t) = \frac{F(t)}{R(t)} = \frac{d\,F(t)}{dt}\cdot\frac{1}{R(t)}$$

Integrating the above equation between 0 to t

$$\int_0^t Z(t)\,dt = \int_0^t \frac{d\,F(t)}{dt}\frac{1}{R(t)}$$

$$= \int_0^t \frac{d\,F(t)}{1 - F(t)} = -\log\left[1 - F(t)\right]_0^t$$

$$= \left[\log R(t)\right]^t$$

$$= -\log R(t) \text{ as } R(0) \text{ and } \log R(0) = 0$$

$$\therefore \quad \boxed{R(t) = e^{-\int_0^t Z(t).dt}}$$

Cumulative Distribution Function

$$F(t) = 1 - R(t) = 1 - e^{-\int_0^t Z(t).dt}$$

Probability density function

$$F(t) = Z(t) . e^{-\int_0^t Z(t).dt}$$

Rearranging the equation and integrating with proper limits, we have,

$$\lambda(t).dt = \frac{-d\,R(t)}{R(t)}$$

or $$\int_0^t \lambda(t)\,dt = -\int_0^R \frac{d\,R(t)}{dt} = -\ln R(t) \quad \text{... (10)}$$

∴ $$\ln R(t) = -\int_0^t \lambda(t).dt \quad \text{... (11)}$$

Initially, at $t = 0$, $R(t) = 1$, we obtain

$$R(t) = \exp\left[-\int_0^t \lambda(t)\,dt\right] \quad \text{... (12)}$$

This is (equation 12) is a general formula for computing reliability $\lambda(t)$ can be any variable and integratable function of time. But, if we specify that $\lambda(t)$ is constant overtime and $\lambda(t) = \lambda$ (say).

The reliability formula becomes

$$R(t) = e^{-\lambda t}$$