Digital Signal Processing

1) Short note on term 'Signal'

✓ A signal is defined as any physical quantity that varies with time, space or any other independent variable or variables.

✓ Here, signal can be a function of time, distance, position, temperature, pressure, etc.,

✓ In electrical system, associated signals are electric current and voltage.

✓ In mechanical system, associated signals are force, speed, torque, etc., and in electronics and telecommunication, associated signals are speech, music, picture & video.

✓ Signal processing ?

⇒ A signal carries information & ~~of~~ objective of signal processing is to extract this information.

⇒ It is concerned with representing signals in mathematical form / terms and extracting the information ~~(carried out)~~ by carrying out the algorithmic operations on the signal.

✓ System ?

⇒ A system may be defined as an integrated unit composed of diverse, interacting structures to perform a desired task.

⇒ Function of a system is to process a given i/p sequence to generate an output sequence.

2) Digital Signals

✓ A digital signal is a signal that represents data as a sequence of discrete values. A digital signal can only take on one value from a finite set of possible values at a given time.

Advantages of digital signals (over an analog signal)

i) Digital signals are more secure, and effect of distortion, noise & interference is much less or negligible as they are less affected.

ii) These signals use low bandwidth

iii) They allow the signals ~~tran~~ for long distance transmission.

iv) By using these signals, we can translate the messages, audio, video into device language.

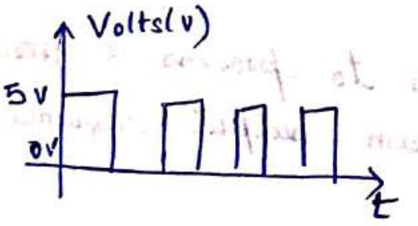
v) A digital signal can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss.

✓ How to represent digital signal?

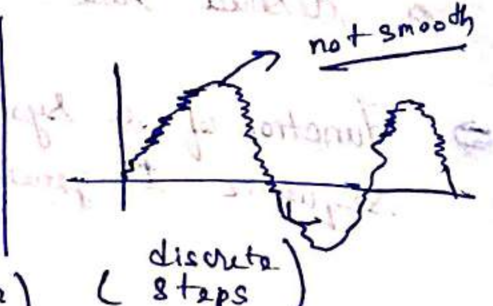
⇒ they have finite set of possible values.

⇒ values can be set anywhere between two values. like either 0V or 5V.

⇒ look like square wave (timing graphs)



(or)



3) Advantages of Digital System (2019) (5 marks)

i) Reproducibility of results and accuracy

ii) Ease of design : No special math skills needed to visualize the behaviour of small digital (logic) circuits.

iii) Flexibility and functionality.

iv) Programmability

v) Speed : A digital logic element can produce an output in less than 10 nano seconds (10^{-9} seconds)

vi) Economy: Due to integration of millions of digital logic elements on a single miniature chip forming low cost integrated circuit (ICs).

4) Advantages of DSP over ASP [check in book Pg - 2]

i) Digital processing is stable, reliable, flexible, predictable and repeatable.

ii) In a digital processor, signals and system coefficients are represented as binary words which enables one to choose any accuracy.

iii) Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time - sharing. This reduces the processing cost, size, weight and maintenance per signal.

iv) With digital filter, linear phase characteristics can be achieved.

v) Multirate processing is possible in digital domain.

vi) Storage of digital data is very easy.

vii) It can process low frequency signals like seismic signals.

(5 marks)

5) Classification of Signals and Systems (2019)

a) Classification of Signals:

i) Single channel and multi-channel signals.
ii) Single dimensional and multi-dimensional signals
iii) Continuous time and discrete time signals.
iv) Analog and digital signals
v) Deterministic and Random signals
vi) Periodic and Non-periodic signals
vii) Causal and Non-causal signals
viii) Even and Odd signal
ix) Energy and Power signal.

Explanation:

i) Single channel and Multi-channel signals

If signal is generated from single source or sensor it is called single channel signal.
If the signals are generated from multiple channels sensors or sources, it is called multi-channels signals. Ex: ECG signals.

ii) Single dimensional (1-D) & Multi-dimensional (M-D)

If a signal is a function of one independent variable it is called as single dimensional signals like speech signal & if signal is function of M independent variables called multi-dimensional signals. Ex: Gray scale level of image or intensity at a particular pixel on black & white TV is an example of M-D signals.

iii) Continuous-time & discrete-time signals

Based on their characteristics (i.e, signals) in the time domain:

Continuous time signals (CTS)	Discrete time signals (DTS)
✓ It can take all values in continuous interval (a, b) where a can be $-\infty$ & b can be ∞.	✓ It is specified only at certain time instants
✓ Its function can be defined continuously in time domain	✓ It is a sequence which is a function defined on positive and negative integers, i.e, $x(n) = \{x(n)\} = \{ x(-1), \underset{\uparrow}{x(0)}, x(1), .\}$ where up-arrow represents sample at $n = 0$.
✓ Described by differential equations	✓ described by difference eq^n.
✓ Denoted by $x(t)$	✓ denoted by $x(n)$
✓ Ex: Speed control of dc motor, sine or exponential wave	✓ μP. & computer based systems use DTS.

✓ Both continuous-time & discrete-time signals are further classified as:

iv) Deterministic signals & non-deterministic signals (or Random signals)

Deterministic signals	Non-deterministic signals
✓ Deterministic signals can be represented or described by mathematical equation. Ex: $x(t) = \alpha t$, ramp signal whose amplitude increases linearly with time slope α. Ex: Sine wave or exponential waveforms.	✓ Random signals that cannot be represented or described by a mathematical equations.
	✓ Ex: Noise signal or speech signal
	✓ Value of random signal cannot be evaluated at any instant of time.
✓ Can be evaluated at time (past, present & future	✓ Its occurrence is random in nature & pattern is irregular.
✓ Nature and amplitude of signal can be predicted at any time	✓ Behaviour of such signal is probabilistic in nature.

vi) Periodic and Non-periodic signal

The signal $x(n)$ is said to be periodic if $\boxed{x(n+N) = x(n)}$ for all n where N is the fundamental period of the signal.

If the signal does not satisfy above property then it is called as non-periodic signals.

For continuous time	For discrete time
Condition for periodicity $\rightarrow$	Condition for periodicity $\rightarrow$
$x(t+T) = x(t), \; -\infty < t < \infty$	$x(n+N) = x(n), \; -\infty < n < \infty$
where T is period of signal	where N is sampling period.
Smallest value of T that satisfies the above condition is called fundamental period T_0, of the signal.	Fundamental period $\Rightarrow T_0 = \frac{2\pi}{\omega}$

Ex: Determine whether the signals are periodic or not

i) $x_1(t) = \sin 15\pi t$ $\Rightarrow \sin(\omega)t \longrightarrow \omega = 15\pi$

$$\Rightarrow T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = \frac{2}{15} = 0.13333333\ldots \text{ seconds}$$

(rational numbers)

ii) $x_6(t) = x_2(t) + x_4(t)$

$$x_6(t) = \sin 20\pi t + \sin 5\pi t$$

$T_2 = 0.1$ seconds $\qquad T_4 = 0.4$ seconds.

Ratio of fundamental frequencies, $\frac{T_2}{T_4} = \frac{0.1}{0.4} = \frac{1}{4}$

can be expressed as ratio of integers.

Hence, $x_6(t)$ is periodic.

vii) ~~Evend~~ Even and Odd Signals

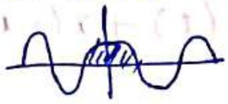
✓ <u>Even signal</u>: If a signal exhibits symmetry in the time domain about the origin, it is called an even signal. It must be identical to its reflection about the origin.

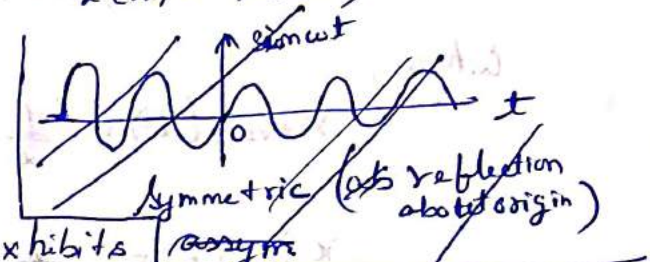
Mathematically, even signal satisfies the relation –

For continuous-time signal, $x(t) = x(-t)$

For discrete " " , $x(n) = x(-n)$

<u>Example</u>: $x_1(t) = \cancel{\sin\omega t}$ $\cos\omega t$





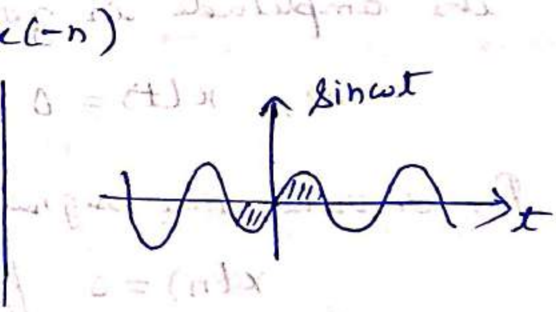
✓ <u>Odd signal</u>: An odd signal exhibits ~~asym~~ anti-symmetry. The signal is not identical to its reflection about the origin, but to its negative.

Mathematically,

For CTS, $x(t) = -x(-t)$

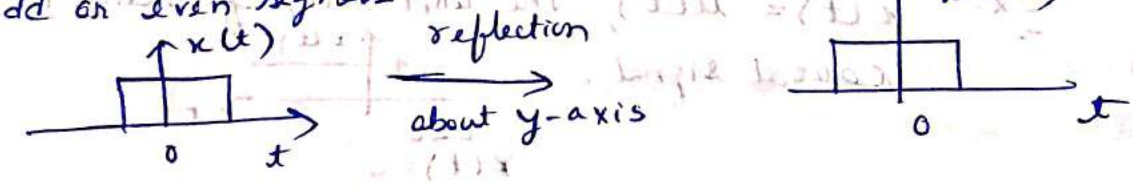
For DTS, $x(n) = -x(-n)$

<u>Example</u>: $x_2(t) = \cancel{\cos\omega t}$ $\sin\omega t$



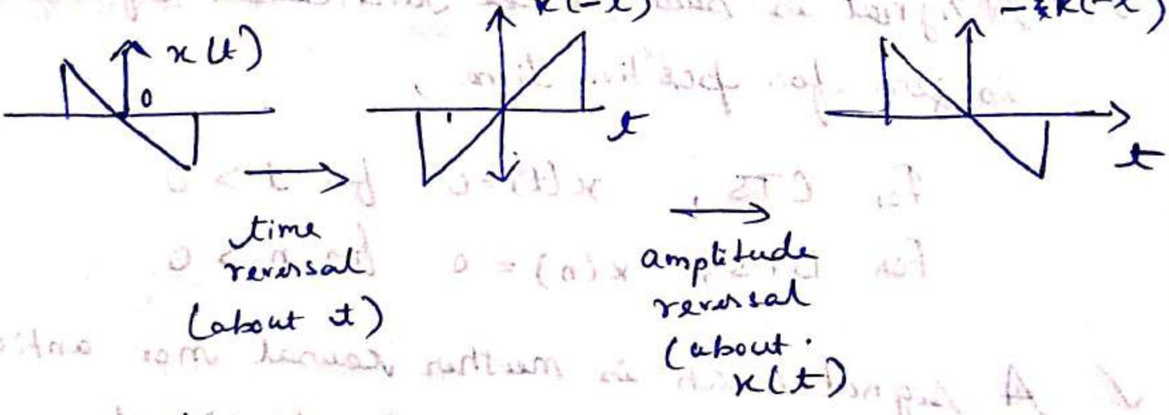
Q) Determine whether they are odd or even signals:

i)



⇒ Even signal

ii)



⇒ odd signal

A signal can be expressed as a sum of two components, namely, the even components of the signal and the odd components of the signal. Even and odd components can be obtained from the signal itself,

$$x(t) = x_{even}(t) + x_{odd}(t)$$

where,

$$x_{even}(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$x_{odd}(t) = \frac{1}{2}\left[x(t) - x(-t)\right].$$

viii) Causal and Non-causal signals

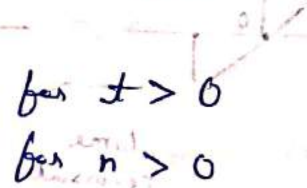
✓ A continuous time signal is said to be causal if its amplitude is zero for negative time.

i.e., $x(t) = 0$ for $t < 0$.

For discrete time signal, condition for causality is

$$x(n) = 0 \quad \text{for } n < 0$$

Ex: $x(t) = u(t)$, the unit step function is causal signal.



✓ A signal is said to be anticausal if its amplitude is zero for positive time,

For CTS, $x(t) = 0$ for $t > 0$

For DTS, $x(n) = 0$ for $n > 0$

✓ A signal which is neither causal nor anticausal is called ~~an~~ a non-causal signal.

ix) Energy and Power Signals

Energy signals: It is one which has finite energy and zero average power.

i.e, $x(t)$ is an energy signal if $0 < E < \infty$, & $P = 0$

✓ It has values only in limited time duration

Ex: Signal having one square pulse, exponentially decreasing has finite energy.

$$E(t) = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E[n] = \lim_{N \to \infty} \sum_{n=-N}^{N} [x(n)]^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Power signals: It is the one which has finite average power and infinite energy, i.e, $0 < P < \infty$, and $E = \infty$.
A power signal is not limited in time, it never ends

Ex: Sine wave in infinite length.

The power of an energy signal is 0, because of dividing finite energy by infinite time (or length).

$$P(t) = \lim_{T \to \infty} \frac{1}{2L} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{E(t)}{2T}$$

$$P(n) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Example: Determine signal energy and signal power for

a) $f(t) = e^{-3|t|}$ b) $f(t) = e^{-3t}$

Sol: (a) Signal Energy,

$$E_\infty = \int_{-\infty}^{\infty} \left[e^{-3|t|}\right]^2 dt$$

$$= \int_{-\infty}^{0} e^{6t} + \int_{0}^{\infty} e^{-6t}\, dt$$

$\int_{-\infty}^{0} e^{-6(-t)}$

$\int_{-\infty}^{0} e^{6t}$

$\int_{0}^{\infty} e^{-6t}$

$$= 2\int_{0}^{\infty} e^{-6t}\, dt$$

$$= -\frac{2}{6}\left[e^{-6t}\right]_{t=0}^{\infty}$$

$$= -\frac{2}{6}[0-1]$$

$$= \frac{1}{3}$$

Signal power, $P_\infty = 0$ since E_∞ is finite.

Hence, the signal $f(t)$ is an energy signal.

(b)
$$E_T = \int_{-T}^{T} \left(e^{-3t}\right)^2 dt$$

$$= \int_{-T}^{T} e^{-6t}\, dt = -\frac{1}{6}\left[e^{-6T} - e^{6T}\right]$$

As $T \to \infty$, E_T approaches infinity.

$$\boxed{E_T \to \infty.}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} E_T = \lim_{T\to\infty} \frac{e^{6T} - e^{-6T}}{12T} = \infty$$

[L' Hospital's Rule]

$$\boxed{P_\infty \to \infty.}$$

Hence, e^{-3t} is neither an ~~eg~~ energy signal nor a power signal.

v) Analog and digital signal

Analog signal	Digital signal
✓ These are basically continuous in time and amplitude signals.	✓ These are basically discrete time signals and discrete amplitude signals.
✓ Examples: ECG signals, Speech ", TV "	✓ These signals are obtained by sampling and quantization process
	✓ All signal representation in digital computers & mobile phones are digital signal.

Discrete time Signals

There are three ways to represent discrete-time signals

1) Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2) Tabular representation, such as

n	....	-2	-1	0	1	2	....
$x(n)$	....	0	0	0	1	4	....

3) Sequence representation

An infinite-duration signal or sequence with time origin $(n=0)$ indicated by the symbol $(\uparrow)$ is,

$$x(n) = \{ \ldots\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0 \ldots\ldots \}$$

A sequence $x(n)$, which is zero for $n<0$, is represented by

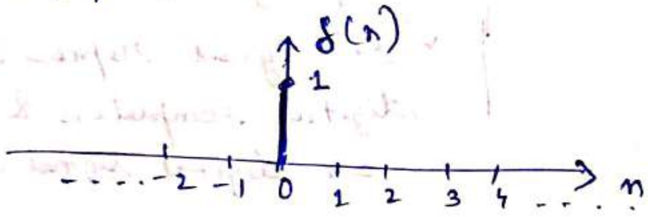
$$x(n) = \{ \underset{\uparrow}{0}, 1, 4, 1, 0, 0 \ldots\ldots \}$$

Elementary Discrete-Time Signals

1) Unit sample signal / Unit impulse signal

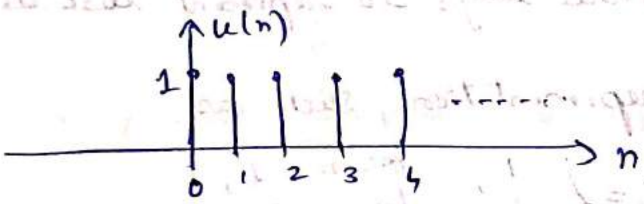
$$\delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0. \end{cases}$$

✓ Unit impulse signal is zero everywhere except at $n=0$, where its value is unity.



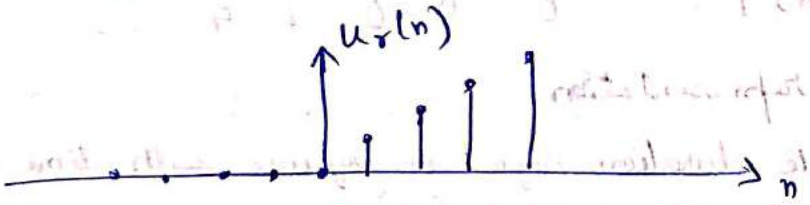
2) Unit step signal

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



3) Unit ramp signal

$$u_r(n) = \begin{cases} n, & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



4) Exponential signal

$$x(n) = a^n \quad \text{for all } n$$

Case: If a is real, $x(n)$ is real signal

If a is complex valued, $a = re^{j\theta}$.

then, $x(n) = r^n e^{j\theta n}$

$= r^n(\cos\theta n + j\sin\theta n)$.

Properties of Discrete - Time Signals (Transformation)

In case of discrete - time signals, the independent variable is the time, n.

i) Shifting

A signal $x(n)$ may be shifted in time, i.e., the signal can be either advanced in the time axis or delayed in the time axis.

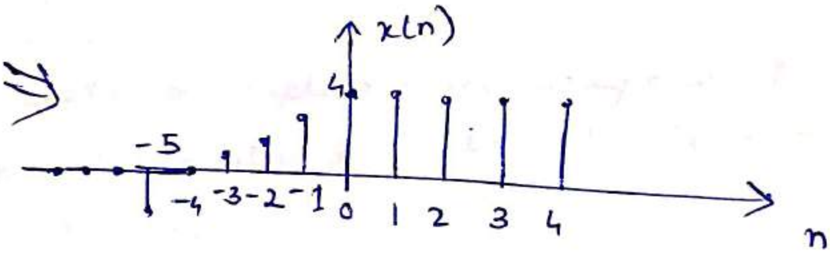
The shifted signal is represented by -

$x(n-k)$, where k is integer.

Case: If 'k' is positive, signal is delayed by k units of time
If 'k' is negative, signal is advanced by k units of time

How to shift? Replace integer 'n' by '$n-k$'.

Original signal: ⇒



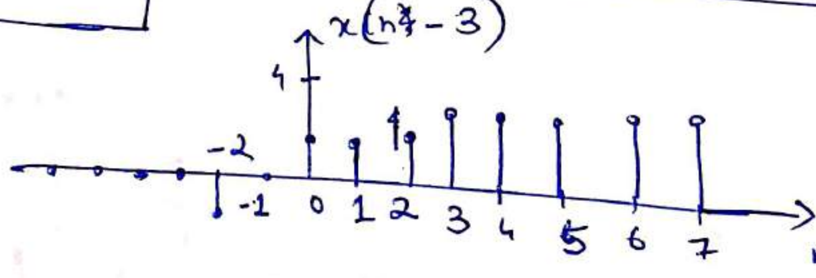
Shift in form $x(n-k)$

i) $x(n-3)$

$k = 3$ (positive)

∴ delay the signal by 3 units

Shift right →

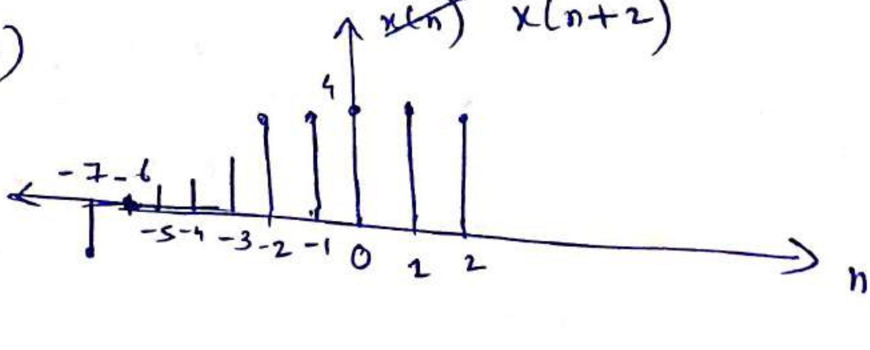


ii) $x(n+2) = x(n-(-2))$

$k = -2$ (negative)

∴ advance by 2 units

Shift left ←



Arrow shifting:

If k is positive, signal is delayed and shift the arrow on left hand side.

If k is negative, " " advanced " " "
" right hand side.

$$x(n) = \{1, -1, 0, 4, -2, \underset{\uparrow \atop n=0}{4}, 0 \ldots\}$$

Delay by 2 samples, $x(n-2)$, $x(n-2) = \{1, -1, 0, \underset{\uparrow \atop n=0}{4}, -2, 4, 0 \ldots\}$

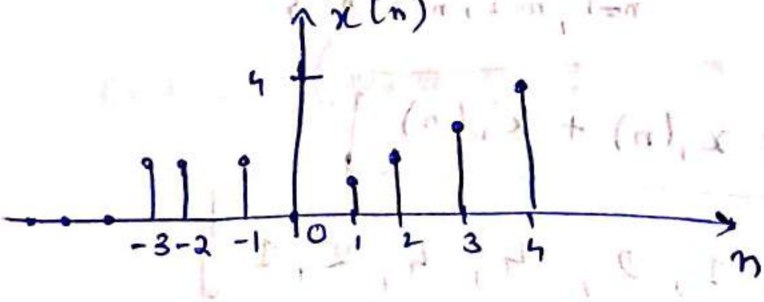
~~Delay to~~

Advanced by 2 samples, $x(n+2) = \{1, -1, 0, 4, -2, 4, 0 \underset{\uparrow}{\ldots}\}$

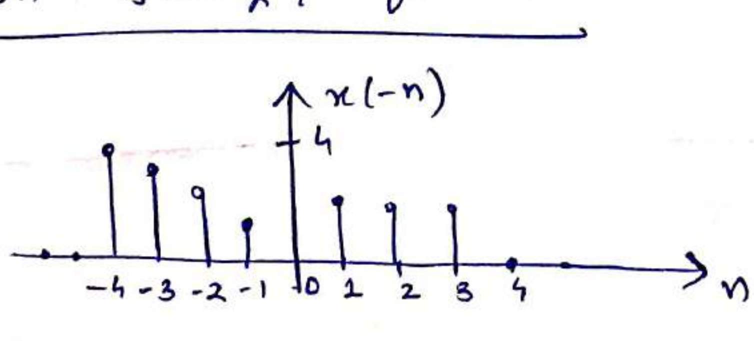
2) Folding / Reflection

This operation is done by replacing the independent variable n by $-n$. This results in folding of the signal about the origin, i.e, $n=0$.

Folding is also known as the reflection of signal about the origin $n=0$.



On folding / reflection



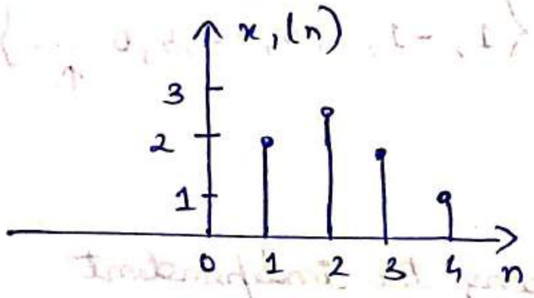
3) <u>Addition:</u>

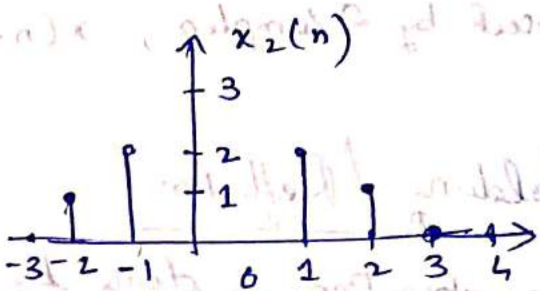
Sum of two signal $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the values of these two signals at that instant,

$$y(n) = x_1(n) + x_2(n) \quad , -\infty < n < \infty$$

✓ Adder generates output sequence which is sum of input sequence.

<u>Ex:</u> $y_1(n) = x_1(n) + x_2(n)$





$$x_1(n) = \{ 2, 3\ 2\ 1 \}$$
$n=1, n=2, n=3, n=4$

$$x_2(n) = \{0, 1, 2, 2, 1, 0, 0\}$$
$n=1, n=2, n=3, n=4$

✓ $\boxed{y_1(n) = x_1(n) + x_2(n)}$

$$y_1(n) = \{0, 1, 2, 4, 4, 2, 1\}.$$

4) Multiplication

Product of two signals:

$$y(n) = x_1(n)\, x_2(n) \quad , \quad -\infty < n < \infty .$$

5) Time scaling

This involves replacing the independent variable n by kn, where k is an integer. This process is called as down sampling.

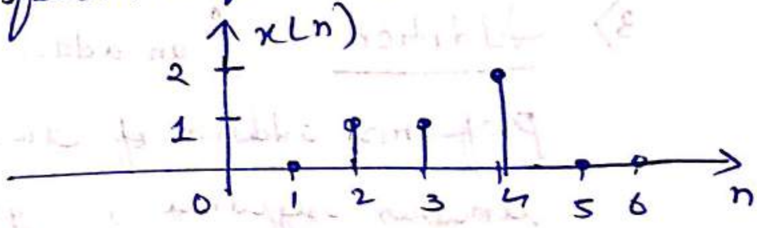
$$y(n) = x(kn) = x(knT) .$$

Sampling rate is changed from $\frac{1}{T}$ to $\frac{1}{kT}$.

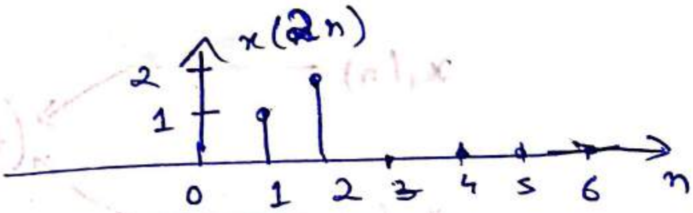
(or) $y(n) = A\, x(m)$

Example: Sketch: i) $x(2n)$ from a given signal $x(n)$ below:

i) $x(n) \longrightarrow$
(given)



$x(2n) \longrightarrow$



$$x(2n) = \{0, 0, \underset{\uparrow}{0}, 1, 2, 0, 0, \dots\}$$

(Block diagram Representation of DTS)

Symbols used in Discrete time system

1) Unit delay

$$x(n) \longrightarrow \boxed{z^{-1}} \longrightarrow y(n) = x(n-1)$$

i/p is $x(n)$ o/p is $x(n-1)$

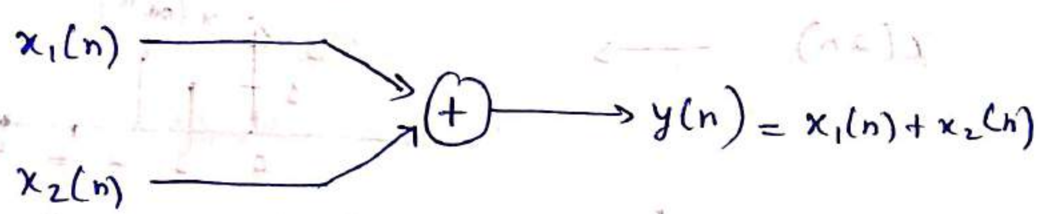
$z^{-1} \to$ denote the unit of delay. [z-transform]

2) Unit advance (moves i/p $x(n)$ ahead by one sample in time)

$$x(n) \longrightarrow \boxed{z^{+1}} \longrightarrow y(n) = x(n+1)$$

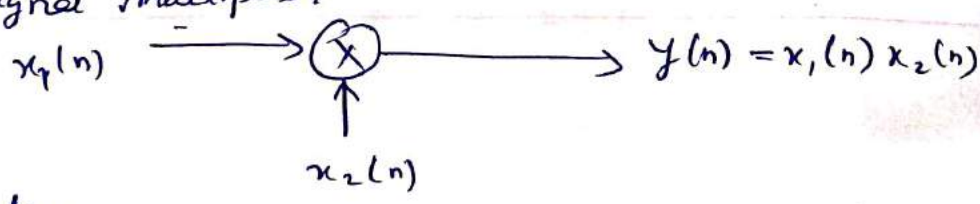
3) Addition (an adder $\oplus$) / memoryless.

performs addition of two signal to form another sequence, $y(n) = x_1(n) + x_2(n)$.



3) Multiplication

⇒ a) Signal multiplier



4) Scaling

⇒ constant multiplier

$$x(n) \xrightarrow{A} y(n) = A\,x(n)$$

Classification of discrete-time systems:

1) Static and Dynamic system

A discrete-time system is called static or memoryless if its output at any instant n depends at most on the i/p sample at the same time, but not on the past or future samples of i/p.

A discrete-time system is called dynamic or to have memory if the output depends upon past or future values of input.

<u>Note</u> ⇒ It is very easy to find out the system is whether static or dynamic by checking the past or future values.

System $y(n)$	
$x(n)$	Static
$x^2(n)$	~~dynamic~~ Static
$x(n^2)$	dynamic
$x(n-2)$	dynamic
$n\,x(n) + x^2(n)$	Static
$x(n) + x(n-2) + x(n+2)$	Dynamic

2) Time invariant vs Time variant Systems

A system is called ~~caller~~ time-invariant if its input-output characteristics do not change with time.

Ex: Thermal noise in electronic components, Printing of documents in a printer.

On the other hand, if its input-output characteristics changes with time then it is called time-variant systems.

Ex: Rainfall per month ⇒ No mathematical analysis.

How to check whether it is time invariant or time variant ?

A relaxed system T is time invariant if and only if

$$x(n) \xrightarrow{T} y(n)$$

implies that

$$x(n-k) \xrightarrow{T} y(n-k)$$

for every input signal $x(n)$ and every time shift k.

Now, delay the input sequence by some amount k & recompute the output,

In general, we can write the output as

$$y(n,k) = T[x(n-k)].$$

Now, if output,

$$y(n,k) = y(n-k),$$

$\Rightarrow$ System is time invariant

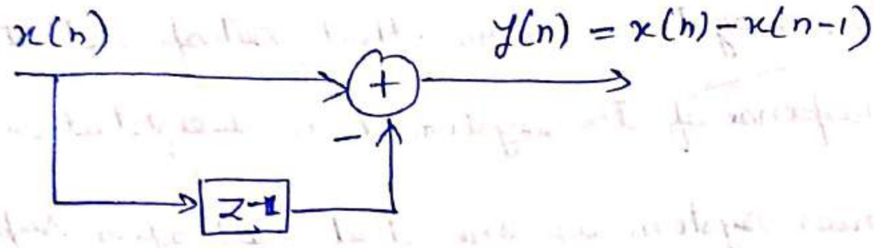
If output,

$$y(n,k) \neq y(n-k),$$

$\Rightarrow$ System is time invariant.

Example: Determine whether the system is time-invariant or time-variant.

a)



System is described by i/p - o/p equations

$$y(n) = T[x(n)] = x(n) - x(n-1) \quad \text{——} \ (1)$$

Now, i/p is delayed by k units in time & applied to system, (block diagram) o/p will be

$$y(n,k) = x(n-k) - x(n-k-1) \quad \text{——} \ (2)$$

If we delay $y(n)$ by k units in time,

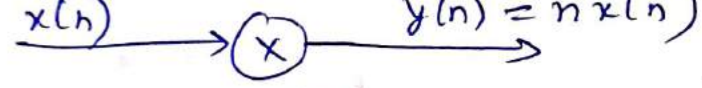
$$y(n-k) = x(n-k) - x(n-k-1) \quad \text{——} \ (3)$$

R.H.S of (2) & (3) are identical.

$$\cancel{y(n,k)} \quad y(n,k) = y(n-k)$$

$\therefore$ The system is time inveriant.

ii)



Input - o/p equation,

$$y(n) = T[x(n)] = n\,x(n) \quad \text{——} \ (1)$$

Response of this system, i/p $x(n)$ is delayed by k units ~~of~~ in time. (replace only $x(n)$ part)

$$\cancel{y(n) = n\,x}$$

$$y(n,k) \ \cancel{y(n)} = n\,x(n-k) \quad \text{——} \ (2)$$

Delay $y(n)$ by k units in time

$$y(n-k) \ \cancel{y(n-k)} = (n-k)\,x(n-k) \quad \text{——} \ (3)$$

$$\therefore \ y(n,k) \neq y(n-k)$$

$\Rightarrow$ Time variant //

3) Linear vs Non-linear systems

~~A linear system is one that satisfies that the response of the system to a weighted sum~~

A linear system is one that satisfies superposition theorem.

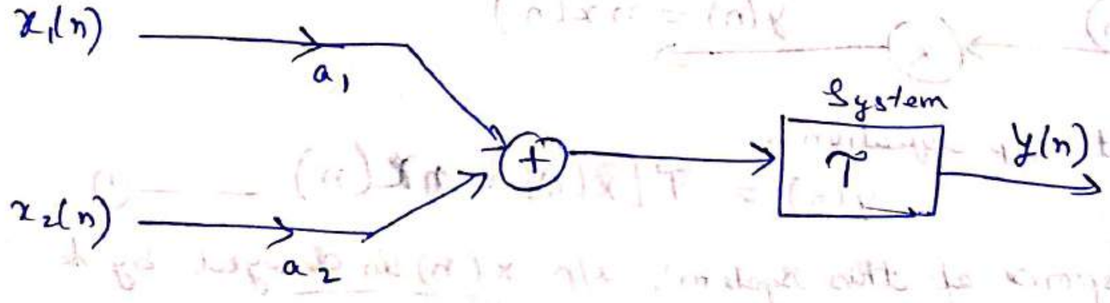
Superposition theorem: It states that the response of the system to a weighted sum of (i/p) signals be equal to corresponding sum of responses (o/p) of the system to each of the individual i/p signals.

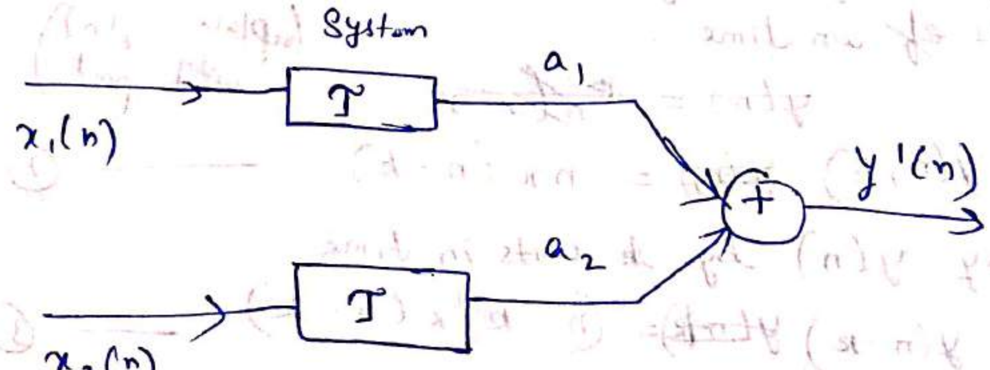
A system is linear if and only if

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

arbitrary i/p sequence $\rightarrow x_1(n)$ & $x_2(n)$

" constants a_1 & a_2





$y(n) = y'(n) \Rightarrow$ Linear

Ex: Check whether linear or non-linear —

i) $y(n) = n x(n)$

Sol:) For two i/p sequences $x_1(n)$ & $x_2(n)$, & corresponding outputs,

$$y_1(n) = n\, x_1(n)$$
$$y_2(n) = n\, x_2(n)$$

Linear combination of two i/p sequences results in o/p

$$y_3(n) = T\{a_1 x_1(n) + a_2 x_2(n)\}$$
$$= n\,[a_1 x_1(n) + a_2 x_2(n)]$$
$$= n\, a_1 x_1(n) + n\, a_2 x_2(n) \quad \text{—①}$$

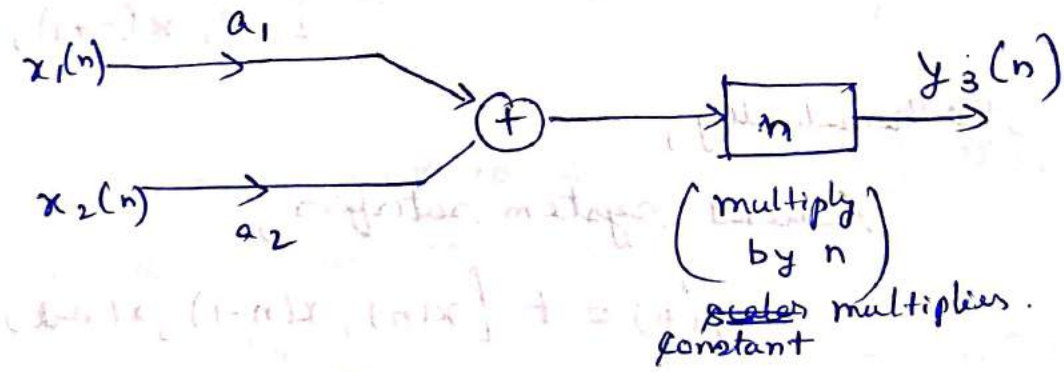
Linear combination of two o/p sequences results in o/p

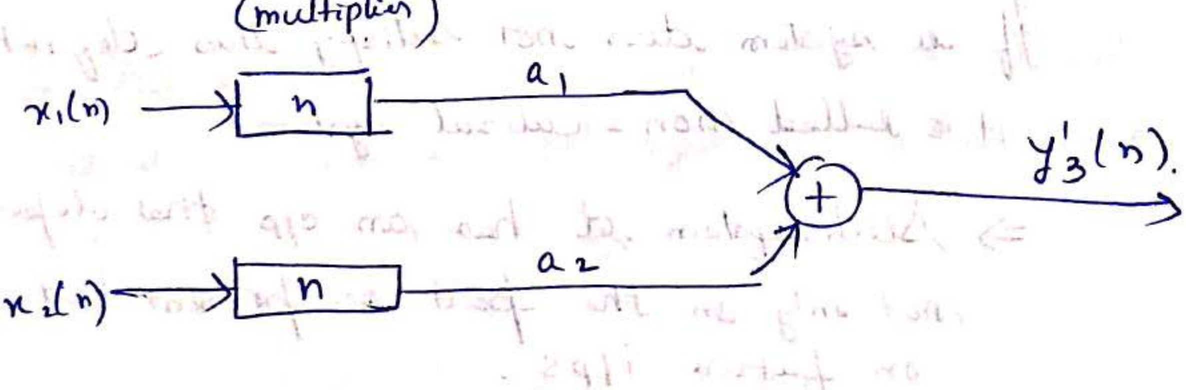
$$y'_3(n) = a_1 y_1(n) + a_2 y_2(n) = a_1 n x_1(n) + a_2 n x_2(n) \quad \text{—②}$$

Since R.H.S of ① & ② are identical

⇒ The system is linear.

Hint :





Ex:

Linear	Non-linear
$n\, x(n)$	$x^2(n)$
$x(n^2)$	$e^{x(n)}$ $\Rightarrow$ if $x(n)=0$, $y(n)=1$.
$x(-n)$	$m\, x(n) + c$
	$\cos[x(n)]$
	$\log 10(\lvert x(n)\rvert)$.

---

4) Causal and Non-causal systems:

A system is said to be causal if the output of the system at any time n [i.e, $y(n)$] depends only on present & past inputs [i.e, $x(n), x(n-1), x(n-2) \ldots$] but does not depend on future inputs [i.e, $x(n+1), x(n+2) \ldots$]

Mathematically,

Causal system satisfies,

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

If a system does not satisfy this definition, it is called non-causal system.

$\Rightarrow$ Such system has an o/p that depends not only on the past or present but also on future i/ps.

$\therefore$ Non-causal satisfies,

$$y(n) = F[x(n), x(n-1), x(n-2), x(n+1) \ldots]$$

Ex: **Causal** | **Non-causal**

Causal	Non-causal
$y(n) = x(n) - x(n-1)$	$y(n) = x(n) + 3x(n+4)$
$y(n) = \sum_{k=-\infty}^{n} x(k)$	$y(n) = x(n^2)$
$y(n) = a\,x(n)$	$y(n) = x(2n)$
	$y(n) = x(-n)$

5) Stable vs Unstable systems

- Stability is an important property that must be considered in any practical applications of the system.
- Since, unstable systems are of extreme behavior and cause overflow in any practical implementation.

Stable system: (Bounded → finite)

BIBO stable: A relaxed system is said to be bounded i/p input - bounded output (BIBO) stable if and only if every bounded sequences produces a bounded output.

The i/p $x(n)$ is said to be bounded if there exists some finite number M_x such that

$$|x(n)| \le M_x < \infty \quad \text{for all } n.$$

If for some bounded i/p sequence $x(n)$, the output is infite) the system is classified as unstable.

Conditions for a system to be BIBO stable:

i) If The system Transfer function is a rational function, The degree of The numerator must be no larger than the degree of the denominator.

ii) The poles of system must like in left half of s-plane or within the unit circle in z-plane.

iii) If a pole lies on Imaginary axis, it must be a single-order one ie, no repeated poles must lie on imaginary axis.

II) Concept of frequency in Discrete Time Signals

N/B

What is frequency? (according to physics)

⇒ Frequency is closely related to a periodic motion called harmonic oscillations, which is described by the sinusoidal functions.

⇒ The concept of frequency is directly related to concept of time. i.e, it has dimesion of inverse time.

$$T = 1/f \text{ or } f = 1/T.$$

A simple harmonic oscillation is methematically described by following discrete-time sinusoidal signals as -

$$\boxed{x(n) = A\cos(\omega n + \theta),} \quad -\infty < n < \infty \qquad (1)$$

Where n is an integer variable called a sample number,

A is the amplitude of sinusoid,

ω is the frequency in radians per sample,

θ is the phase in radians.

If instead of 'ω' we use the frequency variable 'f'

defined by $\boxed{\omega = 2\pi f}$ —— (2)

∴ Relation (1) becomes

$$\boxed{x(n) = A\cos(2\pi f n + \theta)}, \quad -\infty < n < \infty \qquad (3)$$

⇒ In case of sampling of analog sinusoids, we relate frequency variable.

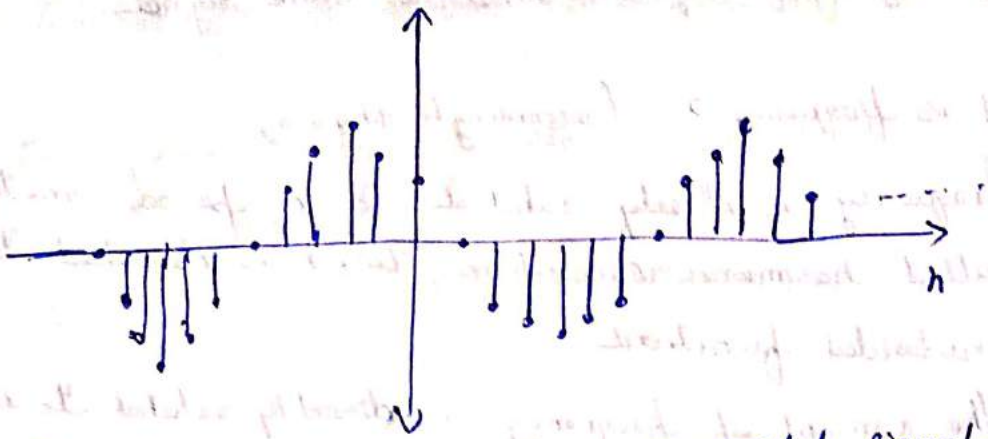


fig: Example of discrete-time sinusoidal signal ($\omega = \pi/6$, and $\theta = \pi/3$).

Properties of discrete-time Sinusoids related to frequency:

1.) A discrete ~~ty~~ time sinusoidal is periodic only if its frequency f is a rational number.

✓ By definition, of periodicity, a discrete time signal is periodic with period N $(N>0)$ if and only if $x(n+N) = x(n)$ for all n — ①

and $N_0 \to$ fundamental period.
→ smallest value of N is fundamental period.

$N \to$ time period of DTS.

✓ For a sinusoid with frequency f_0 to be periodic, we should have

$$\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n+\theta) \quad ②$$

The relation is true if there exists an integer k, such that $2\pi f_0 N = 2k\pi$

$$(\text{or}) \quad f_0 = \frac{k}{N} \quad ③$$

from relation (3), we know that

$\therefore$ A discrete-time sinusoidal signal is periodic only if its frequency f_0 can be expressed as ratio of two integers (i.e, f_0 is rational).

Note

To determine fundamental period N of periodic sinusoid

$\Rightarrow$ First express its frequency f_0 i.e, $f_0 = \frac{k}{N}$

$\Rightarrow$ Cancel the common factors so that k & N are relatively prime.

$\Rightarrow$ Then the fundamental period of sinusoid is equal to N

We can observe that a small change in frequency can result in a large change in the period.

Ex: $f_1 = \frac{31}{60}$ | $f_2 = \frac{30}{60} = \frac{1}{2}$ (In prime no.s.)

$N_1 = 60$ | $N_2 = 2$

---

2) Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.

Ex: $\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$

All sinusoidal sequences,

$$x_k(n) = A\cos(\omega_k n + \theta), \quad k = 0, 1, 2, \ldots$$

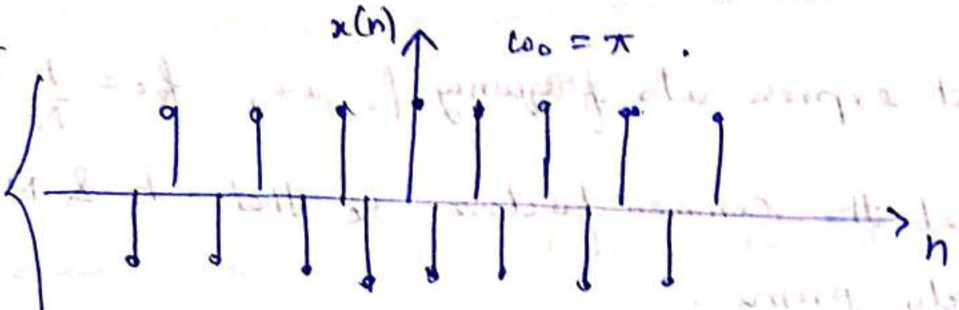
where $\omega_k = \omega_0 + 2k\pi$, $\quad -\pi \leq \omega_0 \leq \pi$.

are identical. //

3) The highest rate of oscillation in a discrete time sinusoid is attained when $\omega = \pi$ or $\omega = -\pi$ or $f = \frac{1}{2}$ or $f = -\frac{1}{2}$.

=> Rate of oscillation increases as the frequency increases.

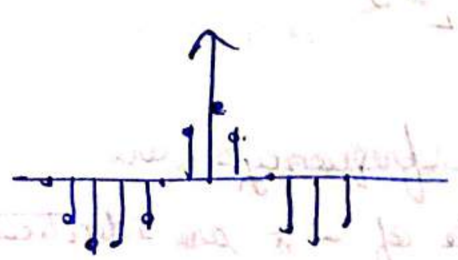
Highest rate at $\omega_0 = \pi$



$x(n) = \cos \omega_0 n$.

and in others:

$\omega_0 = \frac{\pi}{4}$



$\omega_0 = 0$

Analog to Digital And Digital-Analog Conversion

✓ Most of the signals of practical interest such as speech, seismic signals, sonar & various communication signals such as audio & video are analog in nature.

✓ To process analog signal by digital means, it is first necessary to convert them into digital form.

⇒ Convert them to a sequence of numbers having finite precision.

✓ This procedure is called analog-to-digital (A/D) conversion, and the corresponding device are called A/D converters (ADCs).

✓ Analog to digital conversion is a three-step process:

1) Sampling –

This is the first step where a continuous time signal is converted into a ~~dig~~ discrete time signal, obtained by taking "samples" of the continuous-time signal at discrete time instants.

If $x_a(t)$ is the input to the sample, the output is, $x_a(nT) = x(n)$, where T is called sampling interval.

Note: Here $x(n)$ is the discrete-time signal obtained by "taking" samples of analog signal $x_a(t)$ every T seconds.

⇒ Time interval T between successive samples is called sampling interval &

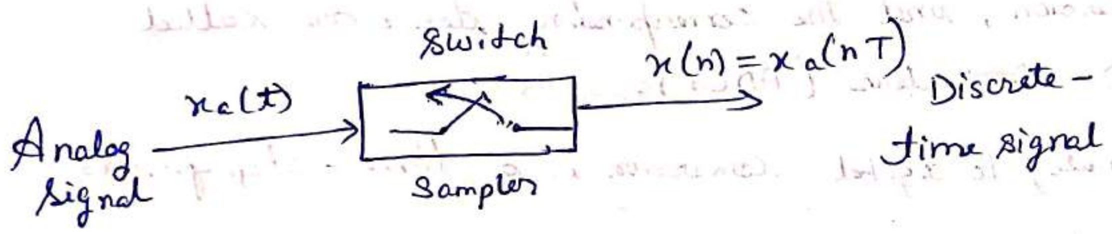
(reciprocal) $\frac{1}{T} = F_s$ = sampling rate (samples per second)
= sampling frequency (in Hz).

✓ Periodic signals establishes a relation between time variables t and n of CTS & DTS.

$$t = nT = \frac{n}{F_s}$$

✓ There also exists a relation between the frequency variable F (or Ω) for analog signals (CTS) and frequency variable f (or ω) for ~~digit~~ discrete-time signals.

$$x_a(t) = A \cos(2\pi F t + \theta)$$



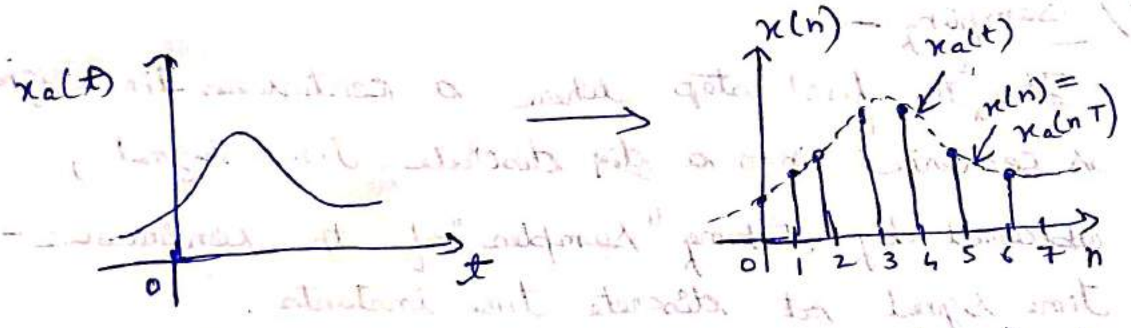


fig: Periodic sampling of analog signal

Relation among Frequency variables.

CTS	DTS
$\Omega = 2\pi F$	$\omega = 2\pi f$
$\frac{\text{radian}}{\text{sec}}$ Hz	$\frac{\text{radians}}{\text{Sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	$-\pi/T \leq \Omega \leq \pi/T$
$-\infty < \Omega < \infty$	$-F_s/2 \leq F \leq F_s/2$
$-\infty < F < \infty$	

DTS

$$-\pi \leq \omega \leq \pi$$

$$-\frac{1}{2} \leq f \leq \frac{1}{2}$$

Relation:

$$\omega = \Omega T, \quad f = F/F_s$$

$$\Omega = \omega/T, \quad F = f/F_s.$$

Highest frequencies in DTS,

$$\omega = \pi, \quad \text{or} \quad f = \frac{1}{2}.$$

Highest values of F & Ω

$$\therefore F_{max} = \frac{F_s}{2} = \frac{1}{2T}$$

$$\Omega_{max} = \pi F_s = \frac{\pi}{T}.$$

2) **Quantization:**

This is the conversion of discrete-time continuous-valued signal into a discrete-time, discrete-valued (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values.

Quantization error – Diffrence between unquantized sample $x(n)$ & quantized sample $x_q(n)$.

3) **Coding:**

In the coding process, each discrete values $x_q(n)$ is represented by a b-bit binary sequence.

A/D conversion:

We modelled the A/D converter as a sampler followed by a quantize and coder

⇒ A/D conversion is performed by a single device that takes $x_a(t)$ and produces a binary-coded number.

⇒ Sampling is always performed before quantization.

Why do we use D/A converters?

Note: In case of speech processing it is desirable to convert the processed digital signals into analog form.

⇒ Obviously, we cannot listen to the sequence of samples representing a speech signal or see the numbers corresponding to a TV signal.

The process of converting a digital signal into an analog signal is known as digital-to analog conversion (D/A converters).

⇒ All digital to analog converters "connect the dots" in a digital signal.

Ex: Simple form of D/A conversion, called a zero-order hold or staircase approximation.

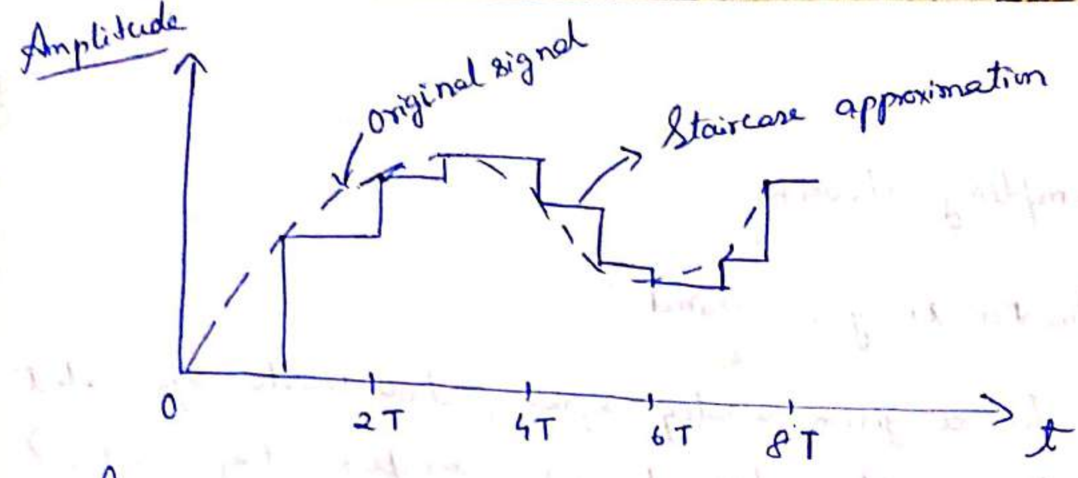


fig: Zero-order hold D/A conversion.

Properties of sampling:

⇒ Does not result in loss of information

⇒ Does not introduce distortion in the signal if the signal bandwidth is finite.

Properties of quantization:

⇒ Non invertible or irreversible process.

⇒ Which results in signal distortion.

Factors affecting the choise of desired accuracy

Amount of distortion is dependent on accuracy & is measured by number of bits in A/D Conversion.

Factors are:

1. Cost
2. Sampling ~~rate~~ rate / Sampling frequency.

⇒ Cost increases with an increase in accuracy / sampling rate.

Sampling theorem

Note: Questions in your mind:

1) For a given analog signal, how would you select the sampling period T or sampling frequency F_s?

⇒ For this, we should first know some frequency range or BW of particular signal.

Say for example,

For speech signal major frequency components falls below 3000 Hz i.e, 3 kHz.

For TV signals, it contains frequency components upto ~~5000Hz~~ i.e, 5 MHz

2) How ~~togather~~ to gather all the information content of such signals like amp, frequencies & phases of frequency components.

⇒ If we know the maximum frequency content of signals (i.e, general class) then we can specify the sampling rate i.e, F_s, in order to ~~Ai~~ convert analog signals to digital signals.

Suppose, analog signals can be represented as sum of sinusoids of different freq, amp or phase,

$$x_a(t) = \sum_{i=1}^{N} A_i \cos(2\pi F_i t + \theta_i)$$

$N \rightarrow$ no. of frequency components.

Suppose, frequencies donot exceed some known frequency i.e, F_{max}.

Ex: $F_{max} = 3000\ Hz$ (Class of speech signal) (Telephone, voice)

$F_{max} = 5\ MHz$ (TV signals).

Highest values of F,

$$F_{max} = \frac{F_s}{2}$$

{ highest freq in analog signal that can be reconstructed ~~wite~~ from sampled one }

⇒ To avoid ambiguities resulting from 'aliasing', we must select the sampling rate to be sufficiently high.

⇒ That is ~~which~~ why, we must ~~sette~~ select $F_s/2$ to be greater than F_{max}.

⇒ Thus to avoid the problem of aliasing, F_s is selected so that

$$\boxed{F_s > 2F_{max}.}$$

⇒ Sampling theorem. //

where F_{max} is the highest frequency component in analog signal.

Nyquist sampling theorem:

If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ & the signal is sampled at the rate $F_s > 2F_{max} = 2\overset{\text{bandwidth}}{B}$, then $x_a(t)$ can be exactly recovered from its sample values using interpolation function,

$$g(t) = \frac{\sin 2\pi Bt}{2\pi Bt}$$

When the sampling of $x_a(t)$ is performed at the minimum sampling rate $F_s = 2B$, the reconstruction formula becomes

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{2B}\right) \frac{\sin 2\pi B(t - n/2B)}{2\pi B(t - n/2B)}$$

The sampling rate $\boxed{F_N = 2B = 2F_{max}}$ is called the Nyquist rate.

Ex: Consider the analog signal

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$$

What is the Nyquist rate for this signal?

Sol: Frequencies,

$$F_1 = \frac{50}{2} = 25\,Hz$$

$$F_2 = \frac{300}{2} = 150\,Hz,$$

$$F_3 = \frac{100}{2} = 50\,Hz.$$

Then; $F_{max} = 150\,Hz$.

$$F_s > 2F_{max} = 300\,Hz.$$

Nyquist rate, $F_N = 2F_{max}$
$= 2 \times 150$
$= 300\,Hz.$ //