Numericals

Part I : Basic Probability Theorem

1. The probability of two events A and B are $P(A) = 0.32$, $P(B) = 0.44$ and $P(A \cup B) = 0.58$
   (i) Are the events mutually exclusive ?
   (ii) Are they independent ?
   (iii) Calculate $P(A|B)$ and $P(B|A)$.

2. The probability of two events A and B are such that $P(A) = 0.4$, $P(A \cup B) = 0.8$; $P(A \cap B) = 0.2$. Determine
   (i) $P(B)$, (ii) $P(A|B)$ (iii) $P(B|A)$

3. For three events A, B and C, if $P(A) = 0.8$, $P(B) = 0.3$, $P(C) = 0.4$, $P(A|B \cap C) = 0.5$ and $P(B|C) = 0.6$
   (i) Determine whether events B and C are independent.
   (ii) Determine whether events B and C are mutually exclusive.

Q1) Given, $P(A) = 0.32$ $P(B) = 0.44$ $P(A \cup B) = 0.58$

(i) for mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

But $0.58 \neq 0.32 + 0.44$

$\therefore$ The events are not mutually exclusive

(ii)
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= 0.32 + 0.44 - 0.58$$
$$= 0.76 - 0.58$$
$$= 0.18$$

for independent events,

$$P(A \cap B) = P(A) \cdot P(B)$$

But $0.18 \neq 0.32 \times 0.44$

So, the events are not independent

(iii)
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.44} = 0.409$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.18}{0.32} = 0.562.$$

Q)2). Given, $P(A) = 0.4$, $P(A \cup B) = 0.8$, $P(A \cap B) = 0.2$.

(i) We know, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \Rightarrow P(B) = P(A \cup B) + P(A \cap B) - P(A)$

$= 0.8 + 0.2 - 0.4$

$= 0.6$

(ii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.2}{0.6} = 0.33.$

(iii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.2}{0.4} = 0.5$

Q) 3). Given,

$$P(A) = 0.8, \quad P(B) = 0.3 \quad P(C) = 0.4$$
$$P(A/B\cap C) = 0.5 \quad \text{and} \quad P(B/C) = 0.6.$$

(i) $$P(B/C) = \frac{P(B\cap C)}{P(C)} = 0.6$$

$$\Rightarrow P(B\cap C) = 0.6 \times 0.4 = 0.24$$
$$\Rightarrow P(B\cap C) = 0.24.$$

For independent events,

$$P(B\cap C) = P(B).P(C)$$
$$\text{But } 0.24 \neq (0.3)\times(0.4)$$

So, B and C are not independent events.

(ii) Since B & C are dependent events, they are mutually exclusive.

(iii) $\cancel{P(A \cap B \cap C)} = \cancel{P(A \cap (B \cap C))}$

$= \cancel{P((A \cap 0.24)}$

$=$

We have

$P(A/B \cap C) = 0.5$

$\Rightarrow P(A/B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)}$

$\Rightarrow P(A \cap B \cap C) = 0.5 \times 0.24$

$= 0.12$ //

and $P(B \cap C/A) = \dfrac{P(B \cap C \cap A)}{P(A)}$

$= \dfrac{0.12}{0.8}$

$= 0.15$ //

4. Two events E_1 and E_2 are such that $P(E_1|E_2) = 0{\cdot}4$, $P(E_1|\bar{E}_2) = 0{\cdot}5$ and $P(E_2) = 0{\cdot}6$. Find the value of $P(E_1)$.

5. The occurrence of a particular event A depends on two mutually exclusive events B_1 and B_2. If $P(B_1) = 0{\cdot}4$ and $P(B_2) = 0{\cdot}2$ then $P(A) = 0{\cdot}34$ but if $P(B_1) = 0{\cdot}2$ and $P(B_2) = 0{\cdot}4$ then $P(A) = 0{\cdot}32$. Evaluate $P(A|B_1)$ and $P(A|B_2)$.

7. Two machines produce the total output of a factory. Machine 1 produces 70% and machine 2 produces 30% of the output. Five percent of the output of machine 1 is defective and 8% from machine 2. If a finished item is selected at random, what is the probability of it being defective?

(Hints: apply total prob. th^{m}) [Ans. 0·059]

(4) given,

$P(E_1/E_2) = 0.4$, $P(E_1/\bar{E}_2) = 0.5$ and $P(E_2) = 0.6$

$P(E_1) = ?$

$P(E_1/E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)}$

$\Rightarrow 0.4 = \dfrac{P(E_1 \cap E_2)}{0.6}$

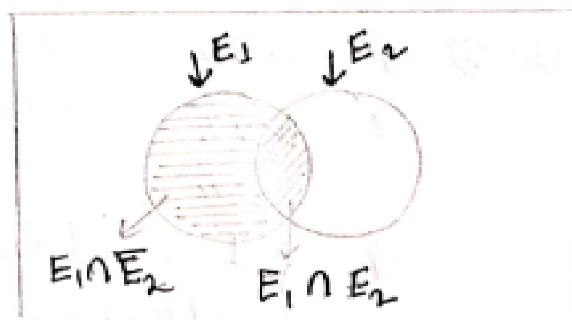
$\Rightarrow P(E_1 \cap E_2) = 0.24$

$P(E_1/\bar{E}_2) = \dfrac{P(E_1 \cap \bar{E}_2)}{P(\bar{E}_2)}$

$\Rightarrow 0.5 = \dfrac{P(E_1 \cap \bar{E}_2)}{0.4}$

$\Rightarrow P(E_1 \cap \bar{E}_2) = 0.2$

$P(\bar{E}_2) = 1 - 0.6$
$= 0.4$



∴ From ~~[illegible]~~ venn diagram

$E_1 = (E_1 \cap \bar{E}_2) + (E_1 \cap E_2)$

$= 0.2 + 0.24$

$= 0.44$

Q)5) If, $P(B_1) = 0.4$, $P(B_2) = 0.2$ then $P(A) = 0.34$.

If, $P(B_1) = 0.2$, $P(B_2) = 0.4$ then $P(A) = 0.32$.

We know

$$\boxed{\sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)}$$

$\Rightarrow P(A \cap B_1) + P(A \cap B_2) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2)$

1st case

$P(A \cap B_1) = P(A/B_1)P(B_1)$
$= P(A/B_1)\, 0.4$

$P(A \cap B_2) = P(A/B_2)P(B_2)$
$= P(A/B_2)(0.2)$

Now,

$P(A) = P(A \cap B_1) + P(A \cap B_2)$

$P(A) = P(A/B_1)\, 0.4 + P(A/B_2)\, 0.2$.

$\Rightarrow 0.34 = P(A/B_1)\, 0.4 + P(A/B_2)\, 0.2$ —— (i)

2nd case

$P(A \cap B_1) = P(A/B_1)0.2 \qquad P(A \cap B_2) = P(A/B_2)0.4$

Now, $P(A) = P(A/B_1)0.2 + P(A/B_2)0.4$

$\Rightarrow 0.32 = P(A/B_1)0.2 + P(A/B_2)0.4$ —— (ii)

(i) − (ii) $\Rightarrow 0.02 = P(A/B_1)0.2 - P(A/B_2)0.2$

$\Rightarrow \dfrac{0.02}{0.2} = P(A/B_1) - P(A/B_2)$

$\Rightarrow 0.1 = P(A/B_1) - P(A/B_2)$

$\Rightarrow P(A/B_1) = 0.1 + P(A/B_2).$

$\therefore$ (i) $\Rightarrow 0.34 = ((0.1) + P(A/B_2))0.4 + 0.2P(A/B_2)$

$\Rightarrow 0.34 = 0.04 + 0.4P(A/B_2) + 0.2P(A/B_2)$

$\Rightarrow 0.34 = 0.04 + 0.6P(A/B_2)$

(4) $\Rightarrow \boxed{P(A/B_2) = 0.5} \quad \boxed{P(A/B_1) = 0.6}$

7) Let,

A = event of defective product

B_1 = event of product made by machine 1

B_2 = " " " " " " 2.

$\therefore$ $P(B_1) = 0.7$ $\quad$ $P(B_2) = 0.3$.

$P(A/B_1) = 0.05$ $\quad$ $P(A/B_2) = 0.08$

$$P(A) = P(A/B_1)\,P(B_1) + P(A/B_2)\,P(B_2)$$

$$P(A) = (0.05)(0.7) + (0.08)(0.3)$$

$$\boxed{P(A) = 0.059}$$

rt II : Binomial Distribution

8. Ten engines undergoing testing. If the failure prob. for an individual engine is 0·10, what is the probability that more than two engines will fail the test?

9. The probability of an engine's failing during a 30-day acceptance test is 0·3 under adverse environmental conditions. Eight engines are included in such a test. What is the probability of the following:

(i) none will fail?

(ii) all will fail?

(iii) more than half will fail?

(3) Given,

$n = 10 \quad p = 0.1 \quad q = 0.9. \quad r = 0, 1, 2.$

~~P(0 defect machine~~ $P(0 \text{ engine fail}) = {}^{10}C_0 (0.1)^0 (0.9)^{10}$

$= 0.3487$

$P(1 \text{ engine fail}) = {}^{10}C_1 (0.1)^1 (0.9)^9$

$= 0.3874$

$P(2 \text{ engine fail}) = {}^{10}C_2 (0.1)^2 (0.9)^8$

$= 0.1937$

$\therefore P(\text{more than 2 engines fail}) = 1 - (0.3487 + 0.3874 + 0.1937)$

$= 0.0702.$

⑨ Given,

$n = 8 \quad p = 0.3 \quad q = 0.7$

(i) $r = 0$

$P(\text{no engine fail}) = {}^8C_0 (0.3)^0 (0.7)^8$

$= 1 \times 1 \times 0.0576$

$= 0.0576.$

(ii) $r = 8$

$P(\text{all engine fail}) = {}^8C_8 (0.3)^8 (0.7)^0$

$= 0.6561 \times 10^{-4}$

(iii) $r > 5, 6, 7, 8$

$P(\text{more than half will fail}) = {}^8C_5 (0.3)^5 (0.7)^3 + {}^8C_6 (0.3)^6 (0.7)^2$

$+ {}^8C_7 (0.3)^7 (0.7)^1 +$

${}^8C_8 (0.3)^8 (0.7)^0$

(6) $= 0.05796765$ //

10. A thermal power plant buys four boilers. If the probability of a boiler functioning without failure for a year is 0·7, plot probability mass function (pmf) and distribution function (cdf) of the status of the boilers at the end of the year.

11. There is a 5% chance of dimensional distortion (considered as failure) in each of the forged connecting rods produced in an engine manufacturing plant.

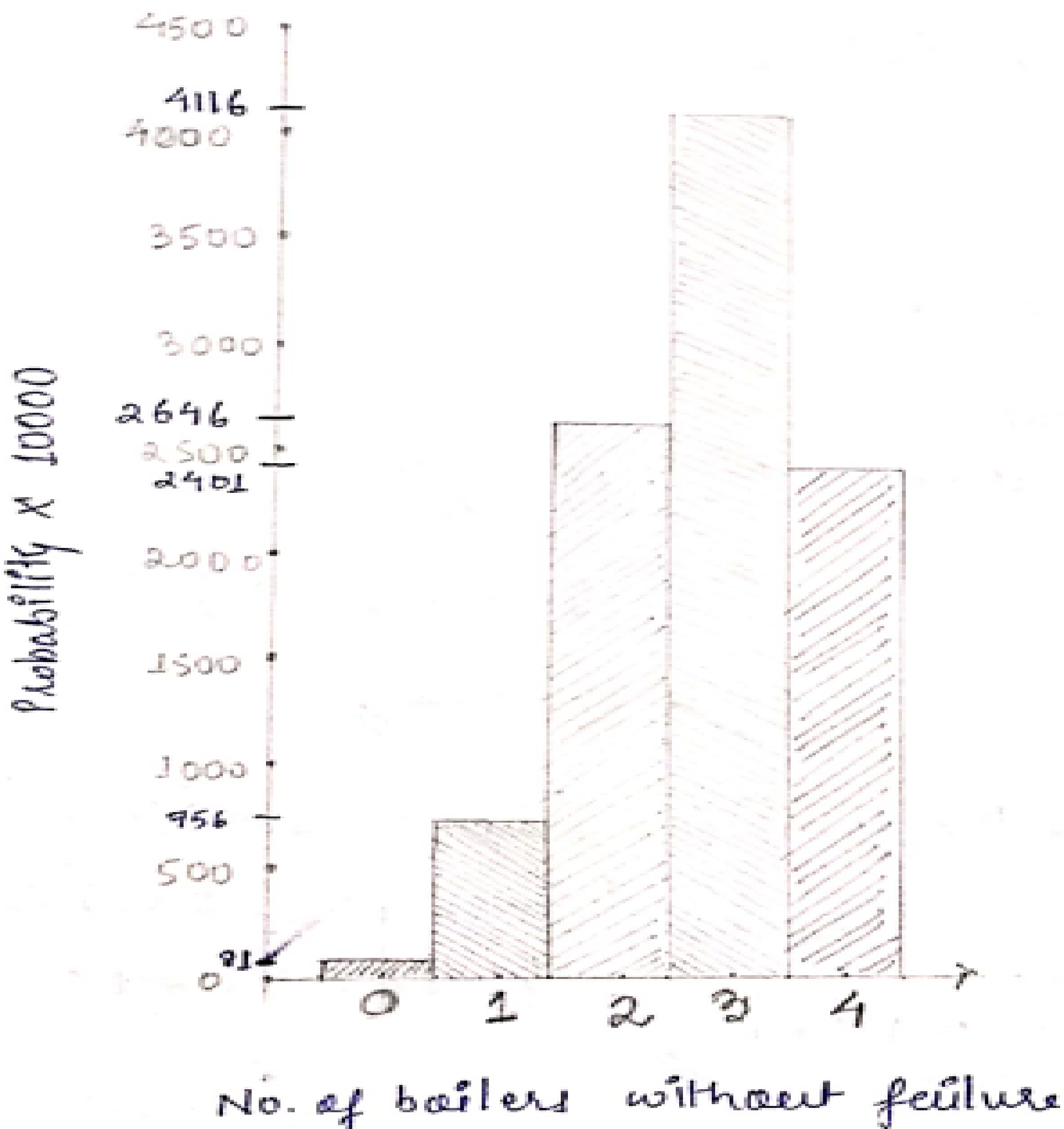
(a) Determine the probability of finding exactly two connecting rods with dimensional distortion when 15 connecting rods are inspected. Assume that the connecting rods are independent w.r.t. dimensional distortions.

(b) Determine the probability that the number of connecting rods with dimensional distortion lies between 2 and 5 when 15 connecting rods are inspected.

[Ans. (a) 0·134752 (b) 0·170899]

(10) Given, $n = 4$ $p = 0.7$ $q = 0.3$

No. of boiler functioning without failure	failed	Individual prob expression	Value	Cumulative probability
0	4	${}^4C_0 (0.7)^0 (0.3)^4$	81/10000	81/10000
1	3	${}^4C_1 (0.7)^1 (0.3)^3$	756/10000	837/10000
2	2	${}^4C_2 (0.7)^2 (0.3)^2$	2646/10000	3483/10000
3	1	${}^4C_3 (0.7)^3 (0.3)^1$	4116/10000	7599/10000
4	0	${}^4C_4 (0.7)^4 (0.3)^0$	2401/10000	10000/10000



Probability x 10000
4500
4116
4000
3500
3000
2646
2500
2401
2000
1500
1000
956
500
81
0
0
1
2
3
4
No. of boilers without failure

Fig → pmf.

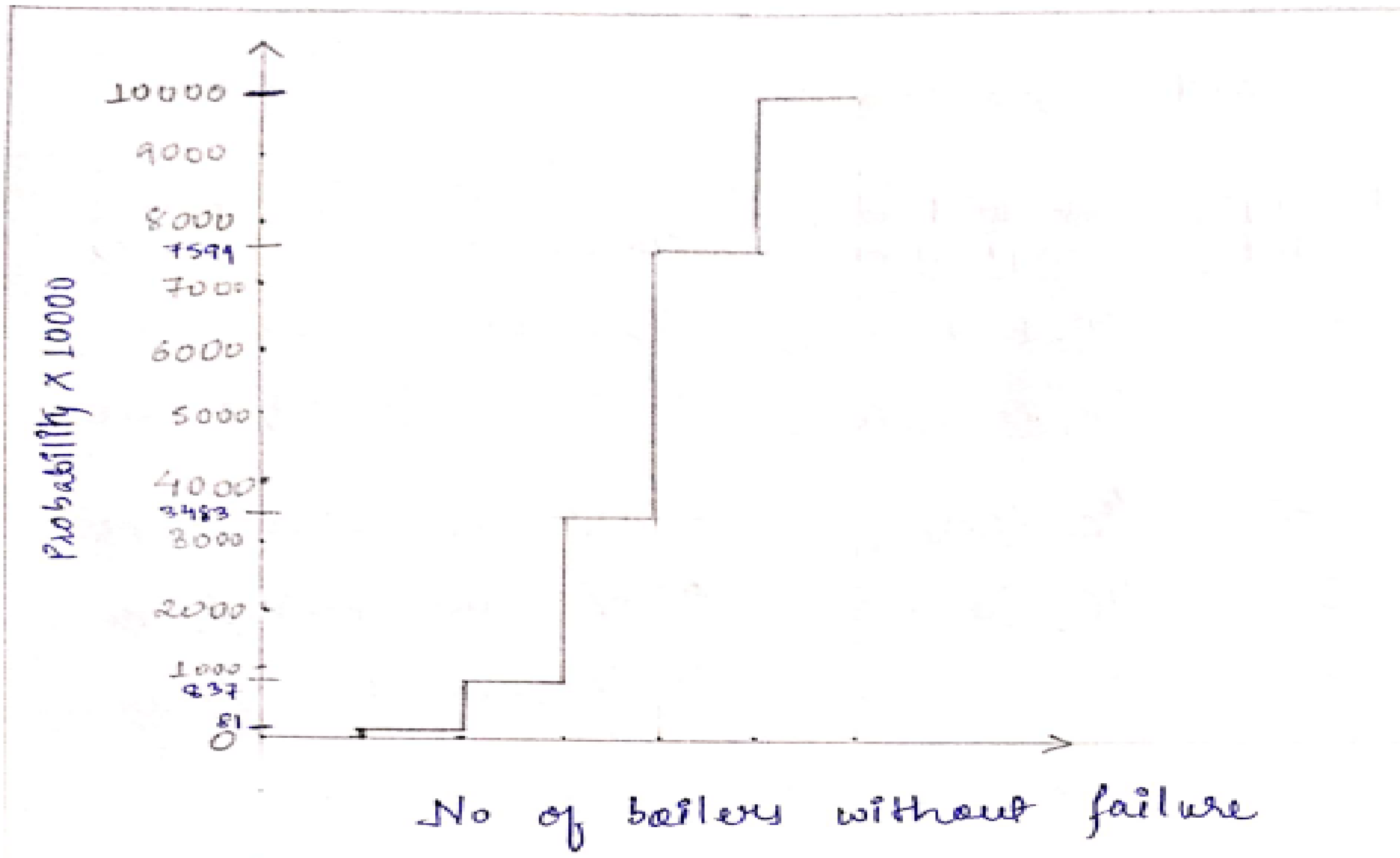

10000
9000
8000
7594
7000
6000
5000
4000
3483
3000
2000
1000
837
51
0
Probability X 10000
No of boilers without failure

Fig → cdf.

11) Given, $n = 15$ $p = 0.05$ $q = 0.95$

(a) P(exactly 2 connecting rods with dimensional distortion) $= {}^{n}C_2 (p)^2 (q)^{n-2}$

$$= {}^{15}C_2 (0.05)^2 (0.95)^{13}$$

$$= 0.134752$$

(b) P($2 \leq$ no. of distortional rods ≤ 5) $= {}^{15}C_2 (0.05)^2 (0.95)^{13} + {}^{15}C_3 (0.05)^3 (0.95)^{12}$

$$+ {}^{15}C_4 (0.05)^4 (0.95)^{11} + {}^{15}C_5 (0.05)^5 (0.95)^{10}$$

$$= 0.134752 + 0.030733 + 0.0048525 + 0.0005618$$

$$= 0.170899$$

PART III : Probability density function & distribution function.

12. Find the probability density function of a random variable X for which the cumulative distribution fn is given by

$$F_X(x) = 1 - e^{-5x}, \quad x > 0$$

13. For the probability density fn

$$f(x) = Kx(1-x), \quad 0 \leq x \leq 1$$
$$= 0, \quad \text{otherwise}.$$

Determine K and mean (μ).

Q) 12) Given,

$$F_X(x) = 1 - e^{-5x} \quad , \quad x > 0.$$

So, $f(x) = \dfrac{d\,F_X(x)}{dx}$

$$= \frac{d}{dx}\left(1 - e^{-5x}\right)$$

$$= 0 + (5)e^{-5x}$$

$$\boxed{f(x) = -5e^{-5x}}$$

(13) given,

$$f(x) = Kx(1-x) \quad , 0 \leq x \leq 1$$
$$= 0 \quad , \text{otherwise}.$$

So, $F(x) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow 1 = \int_0^1 Kx(1-x)\,dx$

$\Rightarrow 1 = K\int_0^1 (x-x^2)\,dx$

$\Rightarrow 1 = K\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$\Rightarrow 1/K = \frac{1}{2} - \frac{1}{3}$

$\Rightarrow 1/K = \frac{1}{6}$

$\Rightarrow \boxed{K = 6}$ //

$\mu(\text{mean}) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^1 x\,Kx(1-x)\,dx.$

$= K\int_0^1 (x^2 - x^3)\,dx$

$= K\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1$

$= K\left(\frac{1}{3} - \frac{1}{4}\right)$

$= K\frac{1}{12}$

$= K/12$

$= 6/12$

$= 1/2$

14. The probability density function of the wind load acting on a water tank is given by

$$f_X(x) = \frac{3x}{50\times10^4}(100-x) \quad ; \quad 0 \le x \le 100 \text{ units}$$

$$= 0 \quad ; \quad \text{otherwise}$$

Find the probability of realizing the wind load to be less than 20 units OR greater than 80 units.

[Ans: 0.208].

15. The diameter of a rod produced on a machine during mass production with the specification of the nominal diameter as 20 mm, is known to follow the following distribution:

$$f_X(x) = 0 \quad ; \quad x < 20 \text{ mm}.$$

$$= 15\, e^{-15(x-20)} \quad ; \quad x \ge 20 \text{ mm}.$$

Determine (a) the probability distribution function of the diameter of the rod. (b) If rods with diameter larger than 20.2 mm are not acceptable, determine the portion of the rods that are accepted.

[Ans (a) $F_X(x) = 0$ for $x < 20$ mm, $F_X(x) = 1 - e^{-15(x-20)}$, $x \ge 20$

(b) 95.0213%]

14) Given,

$$f_x(x) = \frac{3x}{50 \times 10^4}(100 - x) \quad ; \quad 0 \leq x \leq 100 \text{ units}$$

$$= 0 \quad ; \quad \text{otherwise.}$$

Probability of realizing the wind load to be less than 20 units,

$$F(x) = \int_0^{20} f(x)\,dx$$

$$= \int_0^{20} \frac{3x}{50 \times 10^4}(100 - x)\,dx$$

$$= \frac{3}{50 \times 10^4}\left[100 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^{20}$$

$$= \frac{3}{50 \times 10^4}\left[\frac{100 \times 20^2}{2} - \frac{20^3}{3}\right]$$

$$= 0.104.$$

& Probability of realizing the wind load to be greater than 80 units,

$$F(x) = \int_{80}^{100} \frac{3x}{50\times10^4}(100-x)\,dx$$

$$= \frac{3}{50\times10^4}\left[\frac{100x^2}{2} - \frac{x^3}{3}\right]_{80}^{100}$$

$$= \frac{3}{50\times10^4}\left[\left(50\times100^2 - \frac{100^3}{3}\right) - \left(50\times80^2 - \frac{80^3}{3}\right)\right]$$

$$= 0.104$$

$\therefore$ Total Probability $= 0.104 + 0.104$

$= 0.208$

15) given,

$$f_X(x) = 0 \quad ; \; x < 20\text{ mm}$$

$$= 15\, e^{-15(x-20)} \quad ; \; x \geq 20\text{ mm}$$

(incomplete)

ⓐ $F_X(x) = \int_0^{20} f_X(x)\, dx$

$$= \int_0^{20} 0\, dx$$

$\therefore \boxed{F_X(x) = 0}$ for $x < 20$ mm.

$$F_X(x) = \int_{20}^{\infty} 15\, e^{-15(x-20)} dx$$

$$= 15 \int_{20}^{\infty} e^{-15(x-20)} dx$$

$$= 15 \left[\left(-e^{-15(x-20)} \right) \cdot -\frac{15x^2}{2} + 0 \right]_{20}^{\infty}$$

$$= 15 \left[0 - \left(-e^{-15(20-20)} \cdot -\frac{15 \times 20^2}{2} \right) \right]$$

$$= 15 \Big[$$

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