## * Gauss divergence theorem:

It states that if $V$ is the volume bounded by a closed surface $S$ and $\vec{F}$ is a vector point function of position with continuous derivatives, then

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv} \qquad [\operatorname{div}\vec{F} = \nabla\cdot\vec{F}]$$

where $\hat{n}$ is a unit normal vector at any point of $S$.

Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and $S$ is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=0$.

Soln Given, $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

By Divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}F\,dv$$

$$= \iiint_V (\nabla\cdot\vec{F})\,dv$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)dv$$

$$= \iiint_V \left\{\frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)\right\}dv$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z - 2y + y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}dx\int_{y=0}^{1}dy\left[4\frac{z^2}{2} - 2yz + yz\right]_0^1$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left\{2(1)^2 - 2y(1) + y(1)\right\}dy$$

$$= \int_{x=0}^{1} dx \int_{0}^{1} (2-2y+y)\,dy = \int_{x=0}^{1} dx \int_{0}^{1} (2-y)\,dy$$

$$= \int_{0}^{1} dx \left[ 2y - \frac{y^2}{2} \right]_{0}^{1}$$

$$= \int_{0}^{1} \left(2-\frac{1}{2}\right) dx = \int_{0}^{1} \frac{3}{2}\, dx = \frac{3}{2}\int_{0}^{1} dx$$

$$= \frac{3}{2}\left[x\right]_{0}^{1}$$

$$= \frac{3}{2} \quad \#$$

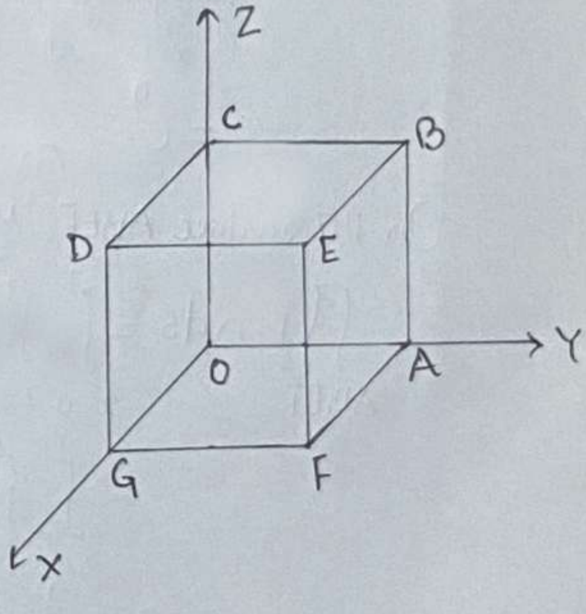
Q. Verify divergence theorem for $\vec{F} = (2x-z)\hat{i} + x^2y\hat{j} - xz^2\hat{k}$ taken over the region bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln. Given, $\vec{F} = (2x-z)\hat{i} + x^2y\hat{j} - xz^2\hat{k}$

By divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$$

Where S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.




Now, from the figure,

On the surface OABC, $x=0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

$$\therefore \iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{1}\int_{y=0}^{1} \vec{F}\cdot\hat{n}\,dy\,dz$$

$$= \int_{0}^{1}\int_{0}^{1} \left\{(2x-z)\hat{i} + x^2y\hat{j} - xz^2\hat{k}\right\}\cdot(-\hat{i})\,dy\,dz$$

$$= \int_{0}^{1}\int_{0}^{1} z\,dy\,dz \quad [\because \text{ in } yz\text{-plane } x=0]$$

$$= \int_0^1 z\,dz\,[y]_0^1 = \int_0^1 z\,dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2} \longrightarrow (1)$$

On the surface DEFG, $x=1$, $n=\hat{i}$ & $ds = dy\,dz$, then

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1}\{(2x-z)\hat{i}+x^2y\hat{j}-xz^2\hat{k}\}\cdot(\hat{i})\,dy\,dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1}(2x-z)\,dy\,dz = \int_{y=0}^{1}\int_{z=0}^{1}(2-z)\,dy\,dz \quad [\because x=1]$$

$$= \int_{y=0}^{1} dy\left[2z-\frac{z^2}{2}\right]_0^1 = \int_0^1\left(2-\frac{1}{2}\right)dy$$

$$= \frac{3}{2}\int_0^1 dy = \frac{3}{2}[y]_0^1$$

$$= \frac{3}{2} \longrightarrow (2)$$

On the surface OGDC, $y=0$, $\hat{n}=-\hat{j}$, $ds = dz\,dx$, then

$$\iint_{OGDC} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{1}\int_{z=0}^{1}\{(2x-z)\hat{i}+x^2y\hat{j}-xz^2\hat{k}\}\cdot(-\hat{j})\,dz\,dx$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} -x^2y\,dz\,dx = \int_{x=0}^{1}\int_{z=0}^{1}(0)\,dz\,dx \quad [\because \text{in } zx\text{-plane } y=0]$$

$$= 0 \longrightarrow (3)$$

On the surface ABEF, $y=1$, $\hat{n}=\hat{j}$ & $ds = dz\,dx$, then

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{1}\int_{z=0}^{1}\{(2x-z)\hat{i}+x^2y\hat{j}-xz^2\hat{k}\}\cdot\hat{j}\,dz\,dx$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} x^2y\,dz\,dx = \int_{x=0}^{1}\int_{z=0}^{1} x^2\,dz\,dx \quad [\because y=1]$$

$$= \int_0^1 x^2\,dx\,[z]_0^1 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3} \longrightarrow (4)$$

On the surface OGFA, $z=0$, $\hat{n}=-\hat{k}$ & $ds=dx\,dy$ then

$$\iint_{OGFA} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{1}\int_{x=0}^{1}\{(2x-z)\hat{i}+x^2y\hat{j}-xz^2\hat{k}\}\cdot(-\hat{k})\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} xz^2\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1}(0)\,dx\,dy \quad [\because \text{in } xy\text{-plane } z=0]$$

$$= 0 \longrightarrow (5)$$

On the surface BCDE, $z=1$, $\hat{n}=\hat{k}$ & $ds=dx\,dy$ then

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{1}\int_{x=0}^{1}\{(2x-z)\hat{i}+x^2y\hat{j}-xz^2\hat{k}\}\cdot\hat{k}\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} -xz^2\,dx\,dy = -\int_{y=0}^{1}\int_{x=0}^{1} x\,dx\,dy \quad [\because z=1]$$

$$= -\int_{y=0}^{1} dy\left[\frac{x^2}{2}\right]_0^1 = -\frac{1}{2}\int_0^1 dy = -\frac{1}{2}[y]_0^1$$

$$= -\frac{1}{2} \longrightarrow (6)$$

Now, adding the eqns (1), (2), (3), (4), (5) & (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \cancel{\frac{1}{2}} + \frac{3}{2} + 0 + \frac{1}{3} + 0 + \left(-\cancel{\frac{1}{2}}\right)$$

$$= \frac{11}{6} \longrightarrow (7)$$

Again, $$\iiint_V \nabla\cdot F\,dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}\cdot\{(2x-z)\hat{i}+x^2y\hat{j}-xz^2\hat{k}\}\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(2+x^2-2xz)\,dx\,dy\,dz$$

$$= \int_0^1 dx\int_0^1 dy\left[2z+x^2z-\cancel{2}x\frac{z^2}{\cancel{2}}\right]_0^1$$

$$= \int_0^1 dx\int_0^1 (2+x^2-x)\,dy$$

$$= \int_0^1 dx \left[2y + x^2y - xy\right]_0^1 = \int_0^1 (2 + x^2 - x)\,dx$$

$$= \left[2x + \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = 2 + \frac{1}{3} - \frac{1}{2}$$

$$= \frac{11}{6} \longrightarrow (8)$$

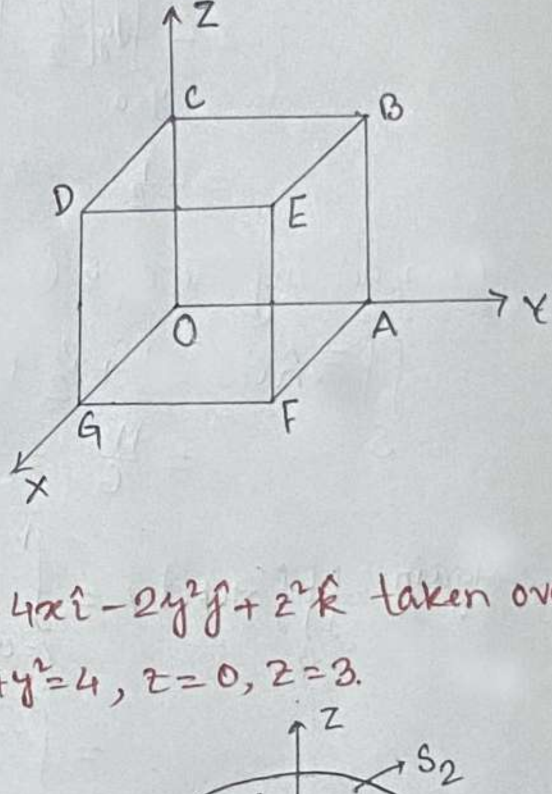
From (7) & (8), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$$

Hence, Gauss's divergence theorem is verified. #

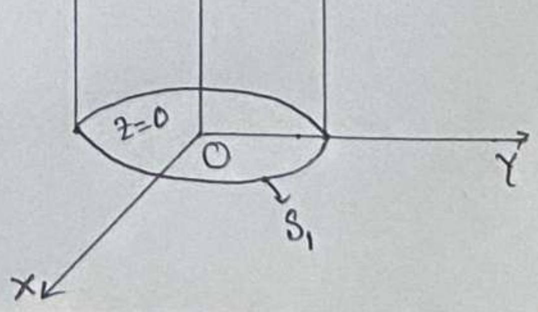
Q. If $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$, verify Divergence theorem and S is the surface of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln. Proceed as above




Q. Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2+y^2=4$, $z=0$, $z=3$.

Soln. By Divergence theorem, we have




$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$$

Here, $\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

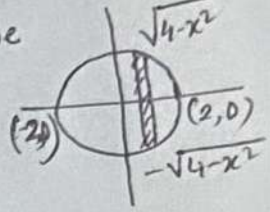
$$= 4 - 4y + 2z$$

⑥

$$\therefore \iiint_V (\nabla\cdot\vec{F})\,dv = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)\,dx\,dy\,dz$$

$\because z = 0 \to 3$

In XY-plane
$x = -2 \to 2$
&
$y = -\sqrt{4-x^2}$
$\downarrow$
$\sqrt{4-x^2}$




$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z-4yz+\cancel{2}\frac{z^2}{\cancel{2}}\right]_0^3 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-12y+9)\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\,dy\,dx \quad \left[\because \int_{-a}^{a} 12y\,dy = 0, \text{ as } 12y \text{ odd function}\right]$$

$$= \int_{-2}^{2} 21\,dx\; 2\int_{0}^{\sqrt{4-x^2}} dy \quad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right]$$

$$= 42\int_{-2}^{2} dx\,[y]_0^{\sqrt{4-x^2}} = 42\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 \quad \left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 84\times 2\sin^{-1}(1) = 84\times \cancel{2}\frac{\pi}{\cancel{2}} \quad [\because \sin(\sin^{-1}\theta)=\theta]$$

$$= 84\pi \longrightarrow ①$$

Again, taking

$S_1$: $x^2+y^2=4$ in $xy$-plane (i.e., $z=0$)

$S_2$: $x^2+y^2=4$ parallel to the $xy$-plane i.e., $z=3$

$S_3$: Curve surface of $x^2+y^2=4$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1}\vec{F}\cdot\hat{n}\,ds + \iint_{S_2}\vec{F}\cdot\hat{n}\,ds + \iint_{S_3}\vec{F}\cdot\hat{n}\,ds$$

On $S_1$: Since $S_1$ is a circle $x^2+y^2=4$ in $xy$-plane so $\hat{n} = -\hat{k}$,
$ds = dx\,dy$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$= \iint_{S_1} z^2\,dx\,dy$$

$$= 0 \quad [\because z = 0] \longrightarrow (2)$$

On $S_2$: Since $S_2$ is a circle $x^2+y^2=4$ in xy-plane so $\hat{n}=\hat{k}$ and $ds = dx\,dy$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} (4x\hat{i} - 2y\hat{j} + z^2\hat{k})\cdot\hat{k}\,dx\,dy = \iint_{S_2} z^2\,dx\,dy$$

$$= 9\iint_{S_2} dx\,dy \quad [\because z = 3]$$

$$= 9 \times \text{area of surface } S_2 = 9(\pi.2^2) = 36\pi \longrightarrow (3)$$

On $S_3$: Since $S_3$ is curve $x^2+y^2=4$. Suppose $f = x^2+y^2-4$, then

$$\text{grad} f = \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-4) = 2x\hat{i} + 2y\hat{j}$$

$$\therefore \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{2(x\hat{i}+y\hat{j})}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{2}$$

Then $$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{S_3} (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right) ds$$

$$= \frac{1}{2}\iint_{S_3} (4x^2 - 2y^3)\,ds = \iint_{S_3} (2x^2 - y^3)\,ds \longrightarrow (4)$$

Since $x^2+y^2=4$ so $x = 2\cos\theta$ & $y = 2\sin\theta$ [$\because$ cylindrical polar coordinates $x = r\cos\theta,\ y = r\sin\theta$ so that $ds = r\,d\theta\,dz$

$\therefore ds = 2\,d\theta\,dz$

From eq$^n$ (4) $\Rightarrow$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \left\{2(2\cos\theta)^2 - (2\sin\theta)^3\right\} 2\,d\theta\,dz$$

$$= 2\int_{\theta=0}^{2\pi}\int_{z=0}^{3} (8\cos^2\theta - 8\sin^3\theta)\,d\theta\,dz$$

$$= \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta\,[z]_0^3$$

$$= 48\int_0^{2\pi} \cos^2\theta\,d\theta - 48\int_0^{2\pi} \sin^3\theta\,d\theta$$

$$= 48\times 2\int_0^{\pi} \cos^2\theta\,d\theta - 48(0)$$

$$= 48\times 2\times 2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 48\times \cancel{4}\times \frac{1}{\cancel{2}}\times \frac{\pi}{\cancel{2}} \quad [\because \text{by Wallis's theorem}]$$

$$= 48\pi \longrightarrow (5) \qquad \left[\because \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, & n \text{ is odd}\end{cases}\right.$$

Adding eqns (2), (3) & (5), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 + 36\pi + 48\pi = 84\pi \longrightarrow (6)$$

From eqn (1) & (6) verifies divergence theorem

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$$

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