§ Volume Integrals:

Any integral which is to be evaluated over a volume is called a volume integral.

Let $\vec{F}$ be a vector point function and volume 'V' enclosed by a closed surface. Then the triple integrals or volume integral is

$\boxed{\iiint_V \vec{F}\,dv}$ → vector point function and $\boxed{\iiint_V \phi\,dv}$ → Scalar point function $\boxed{dV = dx\,dy\,dz}$

Q. If $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla\cdot\vec{F}\,dv$, where V is bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

Soln. Given, $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$

$$\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}\right]$$
$$= \frac{\partial}{\partial x}(2x^2-3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x)$$
$$= 4x - 0 - 2x - 0$$
$$= 2x$$

$$\therefore \iiint_V \nabla\cdot\vec{F}\,dv = \iiint_V 2x\,dx\,dy\,dz$$

$\because y=0, z=0 \Rightarrow 2x=4 \Rightarrow x=2$

$z=0 \Rightarrow 2x+2y=4$
$\Rightarrow 2(x+y)=4$
$\Rightarrow x+y=2$
$\Rightarrow y=2-x$
& $z = 4-2x-2y$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dx\,dy\,dz$$
$$= \int_0^2\int_0^{2-x} 2x\,[z]_0^{4-2x-2y}\,dx\,dy$$
$$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\,dx\,dy$$
$$= \int_0^2\int_0^{2-x} (8x-4x^2-4xy)\,dx\,dy$$
$$= \int_0^2 \left\{(8x-4x^2)[y]_0^{2-x} - 4x\left[\frac{y^2}{2}\right]_0^{2-x}\right\}dx$$

$$= \int_0^2 \left\{ (8x-4x^2)(2-x) - 2x(2-x)^2 \right\} dx$$

$$= \int_0^2 \left\{ 16x - 8x^2 - 8x^2 + 4x^3 - 2x(4-4x+x^2) \right\} dx$$

$$= \int_0^2 (4x^3 - 16x^2 + 16x - 8x + 8x^2 - 2x^3) \, dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x) \, dx$$

$$= \left[2\frac{x^4}{4} - 8\frac{x^3}{3} + 8\frac{x^2}{2} \right]_0^2$$

$$= 2\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2 \right]_0^2$$

$$= 2\left(\frac{16}{4} - \frac{4}{3} \times 8 + 8 \right)$$

$$= \frac{8}{3} \ \#$$

Q. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F} dV$, where V is the region bounded by the surface $x=0, y=0, x=2, y=4, z=x^2, z=2$.

Soln. Given, $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$

$$\therefore \iiint_V \vec{F} dV = \iiint_V (2z\hat{i} - x\hat{j} + y\hat{k}) \, dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k}) \, dz$$

$$= \int_0^2 dx \int_0^4 dy \left[2 \cdot \frac{z^2}{2}\hat{i} - xz\hat{j} + yz\hat{k} \right]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy \left[(4\hat{i} - 2x\hat{j} + 2y\hat{k}) - (x^4\hat{i} - x^3\hat{j} + x^2y\hat{k}) \right]$$

$$= \int_0^2 dx \int_0^4 (4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}) \, dy$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + 2\cdot\frac{y^2}{2}\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - x^2\cdot\frac{y^2}{2}\hat{k} \right]_0^4$$

$$= \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right) dx$$

$$= \left[16x\hat{i} - \overset{4}{\cancel{8}}\cdot\frac{x^2}{\cancel{2}}\hat{j} + 16x\hat{k} - 4\cdot\frac{x^5}{5}\hat{i} + \cancel{4}\cdot\frac{x^4}{\cancel{4}}\hat{j} - 8\cdot\frac{x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - \cancel{16\hat{j}} + 32\hat{k} - \frac{4}{5}\times 32\hat{i} + \cancel{16\hat{j}} - \frac{8}{3}\times 8\hat{k}$$

$$= 32\hat{i} - \frac{128}{5}\hat{i} + 32\hat{k} - \frac{64}{3}\hat{k}$$

$$= \frac{32}{15}\left(3\hat{i} + 5\hat{k}\right) \#$$

Q. Evaluate $\iiint_V \phi \, dV$, where $\phi = 45x^2y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$

Soln Here, $\phi = 45x^2y$

$4x + 2y + z = 8 \quad [y = 0, z = 0]$
$\Rightarrow x = 0 \rightarrow 2$
$4x + 2y = 8 \quad [z = 0]$
$\Rightarrow 2x + y = 4$
$\Rightarrow y = 4 - 2x$
$\therefore y = 0 \rightarrow 4 - 2y$
$\&\ z = 0,\ z = 8 - 4x - 2y$

$$\therefore \iiint_V \phi \, dV = \iiint_V 45x^2y \, dx\,dy\,dz$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} x^2y \, dx\,dy\,dz$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2y \, dx\,dy \left[z\right]_0^{8-4x-2y}$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2y\left(8 - 4x - 2y\right) dx\,dy$$

$$= 90 \int_0^2 \int_0^{4-2x} \left(4x^2y - 2x^3y - x^2y^2\right) dx\,dy$$

$$= 90 \int_0^2 dx \left[4x^2\frac{y^2}{2} - 2x^3\frac{y^2}{2} - x^2\frac{y^3}{3}\right]_0^{4-2x}$$

$$= 90 \int_0^2 dx \left[2x^2y^2 - x^3y^2 - \frac{x^2}{3}y^3\right]_0^{4-2x}$$

$$= 90\int_0^2 dx\left[2x^2(4-2x)^2 - x^3(4-2x)^2 - \frac{x^2}{3}(4-2x)^3\right]$$

$$= 90\int_0^2 \left\{(4-2x)^2(2x^2-x^3) - \frac{x^2}{3}(64-96x+48x^2-8x^3)\right\}dx$$

$$= 90\int_0^2 \left\{(16-16x+4x^2)(2x^2-x^3) - \frac{8}{3}(8x^2-12x^3+6x^4-x^5)\right\}dx$$

$$= 90\int_0^2 \left\{(32x^2-16x^3-32x^3+16x^4+8x^4-4x^5) - \frac{8}{3}(8x^2-12x^3+6x^4-x^5)\right\}dx$$

$$= 90\int_0^2 \left\{(32x^2-48x^3+24x^4-4x^5) - \frac{8}{3}(8x^2-12x^3+6x^4-x^5)\right\}dx$$

$$= 90\int_0^2 \left\{4(8x^2-12x^3+6x^4-x^5) - \frac{8}{3}(8x^2-12x^3+6x^4-x^5)\right\}dx$$

$$= 90\int_0^2 (8x^2-12x^3+6x^4-x^5)\left(4-\frac{8}{3}\right)dx$$

$$= 90\times\frac{4}{3}\int_0^2 (8x^2-12x^3+6x^4-x^5)\,dx$$

$$= 120\left[8\cdot\frac{x^3}{3} - 12\frac{x^4}{4} + 6\frac{x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= 120\left[\frac{8}{3}(2)^3 - 3(2)^4 + \frac{6}{5}(2)^5 - \frac{1}{6}(2)^6\right]$$

$$= 120\left(\frac{64}{3} - 48 + \frac{192}{5} - \frac{\cancel{64}^{32}}{\cancel{6}_3}\right)$$

$$= 120\times\left(\frac{320-720+576-160}{15}\right)$$

$$= \cancel{120}^{8}\times\frac{16}{\cancel{15}}$$

$$= 128$$

#

* Green's Theorem (for a plane):

If $M(x,y)$ and $N(x,y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be continuous functions over a region R bounded by simple closed curve 'C' in xy-plane, then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where 'C' is traversed in the counter clockwise direction.

Note: Green's theorem in vector form

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla \times \vec{F})\cdot \hat{k}\, dR$$

where $\vec{F} = M\hat{i} + N\hat{j}$, $\vec{r} = x\hat{i} + y\hat{j}$, $\hat{k}$ is a unit vector along z-axis and $dR = dx\,dy$.

Q. Verify Green's theorem in the plane for $\oint_C (2xy - x^2)dx + (x+y^2)dy$ where C is the closed curve of the region bounded by $y = x^2$ & $y^2 = x$.

Soln. The given plane is

$$\oint_C (2xy - x^2)dx + (x+y^2)dy \longrightarrow (1)$$

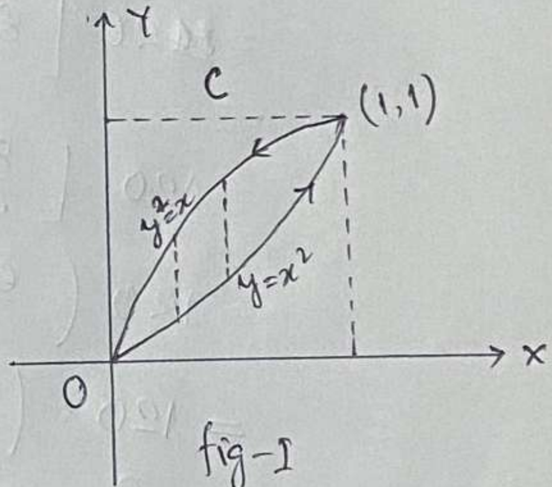


fig-I

The plane curves $y = x^2$ & $y^2 = x$ intersect at $(0,0)$ and $(1,1)$ as shown in fig-I.

Along $y = x^2$, then the line integral (1)

$$= \int_{x=0}^{1} \{2x(x^2) - x^2\}dx + \{x + (x^2)^2\}dx^2$$

$$= \int_0^1 (2x^3 - x^2)dx + \int_0^1 (x + x^4)\,dx^2 \longrightarrow (2)$$

Now, $\int_0^1 (2x^3 - x^2)dx = \left[2\cdot\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \longrightarrow (3)$

And $\int_0^1 (x+x^4)dx^2 = \int_0^1 (\sqrt{t}+t^2)dt$ [putting $x^2 = t$, when $x=0, t=0$ & $x=1, t=1$

$= \int_0^1 (t^{1/2}+t^2)\,dt$

$= \left[\frac{t^{3/2}}{3/2} + \frac{t^3}{3}\right]_0^1 = \left[\frac{2}{3}t^{3/2} + \frac{1}{3}t^3\right] = \frac{2}{3} + \frac{1}{3}$

$= 1 \longrightarrow (4)$

Using (3) & (4) in eq^n (2), we get

$\int_0^1 (2x^3 - x^2)dx + \int_0^1 (x+x^4)dx^2 = \frac{1}{6} + 1 = \frac{7}{6} \longrightarrow (5)$

Along $y^2 = x$, then the line integral (1)

$= \int_{y=1}^{0} \left[\{2y^2\cdot y - (y^2)^2\}dy^2 + (y^2+y^2)dy\right]$

$= \int_1^0 (2y^3 - y^4)dy^2 + 2\int_1^0 y^2 dy \longrightarrow (6)$

Now, $\int_1^0 (2y^3 - y^4)dy^2 = \int_1^0 (2z\cdot z^{1/2} - z^2)dz$ [putting $y^2 = z$, when $y=1, z=1$ & $y=0, z=0$

$= \int_1^0 (2z^{3/2} - z^2)dz = \left[2\cdot\frac{z^{5/2}}{5/2} - \frac{z^3}{3}\right]_1^0 = 0 - \left(\frac{4}{5} - \frac{1}{3}\right)$

$= \frac{1}{3} - \frac{4}{5} = \frac{5-12}{15} = \frac{-7}{15} \longrightarrow (7)$

& $2\int_1^0 y^2 dy = 2\left[\frac{y^3}{3}\right]_1^0 = \frac{2}{3}(0-1) = -\frac{2}{3} \longrightarrow (8)$

Using (7) & (8) in eq^n (6), we get

$\int_1^0 (2y^3 - y^4)dy^2 + 2\int_1^0 y^2 dy = -\frac{7}{15} - \frac{2}{3} = \frac{-7-10}{15} = \frac{-17}{15} \longrightarrow (9)$

$\therefore$ The reqd integral $= \frac{7}{6} + \left(\frac{-17}{15}\right) = \frac{1}{30}$

Again, $\iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint\limits_R (1-2x)\,dx\,dy$

Here, $M = 2xy - x^2$

$N = x + y^2$

$\frac{\partial M}{\partial y} = 2x$

$\frac{\partial N}{\partial x} = 1$

$$= \int_{x=0}^{1}\int_{y=x^2}^{y=\sqrt{x}} (1-2x)\,dx\,dy$$

$\left[\because y = x^2,\ y^2 = x \Rightarrow y = \sqrt{x}\right]$

$$= \int_{x=0}^{1} dx\left[y - 2xy\right]_{x^2}^{\sqrt{x}} = \int_0^1 dx\,(1-2x)\left[y\right]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 (1-2x)(\sqrt{x} - x^2)\,dx = \int_0^1 (1-2x)(x^{1/2} - x^2)\,dx$$

$$= \int_0^1 \left(x^{1/2} - x^2 - 2x^{3/2} + 2x^3\right) dx$$

$$= \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3} - 2\frac{x^{5/2}}{5/2} + 2\,\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - \frac{4}{5} + \frac{1}{2}$$

$$= \frac{1}{30}$$

$$\therefore \oint_C (M\,dx + N\,dy) = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

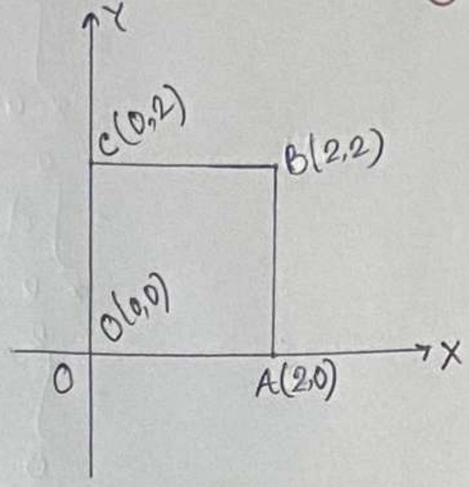
Hence, Green's theorem is verified. #

Q. Verify the Green's theorem in the plane for $\oint_c (x^2 - 2xy)\,dx + (y^2 - x^3y)\,dy$, where 'c' is a square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

Soln. Here, $M = x^2 - 2xy$

$N = y^2 - x^3 y$

$\therefore \dfrac{\partial M}{\partial y} = -2x$ & $\dfrac{\partial N}{\partial x} = -3x^2 y$



Now,

$$\oint_C M dx + N dy = \oint_C (x^2 - 2xy)dx + (y^2 - x^3 y)dy$$

$$= \int_{OA} (x^2 - 2xy)dx + (y^2 - x^3 y)dy + \int_{AB} (x^2 - 2xy)dx + (y^2 - x^3 y)dy$$

$$+ \int_{BC} (x^2 - 2xy)dx + (y^2 - x^3 y)dy + \int_{CO} (x^2 - 2xy)dx + (y^2 - x^3 y)dy$$

Along the line OA, $y = 0$, x varies from 0 to 2 [$dy = 0$]

" " " AB, $x = 2$, y " " 0 to 2 $dx = 0$

" " " BC, $y = 2$, x " " 2 to 0 $dy = 0$

" " " CO, $x = 0$, y " " 2 to 0. $dx = 0$

$$\therefore \oint_C M dx + N dy = \int_0^2 x^2 dx + \int_0^2 (y^2 - 8y)dy + \int_2^0 (x^2 - 4x)dx + \int_2^0 y^2 dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{y^3}{3} - 8\frac{y^2}{2}\right]_0^2 + \left[\frac{x^3}{3} - 4\frac{x^2}{2}\right]_2^0 + \left[\frac{y^3}{3}\right]_2^0$$

$$= \frac{8}{3} + \left(\frac{8}{3} - 16\right) + 0 - \left(\frac{8}{3} - 8\right) + \left(0 - \frac{8}{3}\right)$$

$$= \cancel{\frac{8}{3}} + \cancel{\frac{8}{3}} - 16 - \cancel{\frac{8}{3}} + 8 - \cancel{\frac{8}{3}}$$

$$= -8$$

Now,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left(-3x^2 y + 2x\right) dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x - 3x^2 y)\, dx\, dy$$

$$= \int_{x=0}^{2} dx \left[2xy - 3x^2 \frac{y^2}{2}\right]_0^2$$

$$= \int_0^2 \left\{4x - \frac{3}{2}x^2(4)\right\} dx = \int_0^2 (4x - 6x^2)\, dx$$

$$= \left[4\frac{x^2}{2} - 6\frac{x^3}{3}\right]_0^2 = \left[2x^2 - 2x^3\right]_0^2$$

$$= 8 - 16$$

$$= -8$$

$$\therefore \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Hence, the Green's theorem is verified. #

Example 1. *Verify Green's theorem in the plane for* $\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ *where C is the boundary of the region defined by*

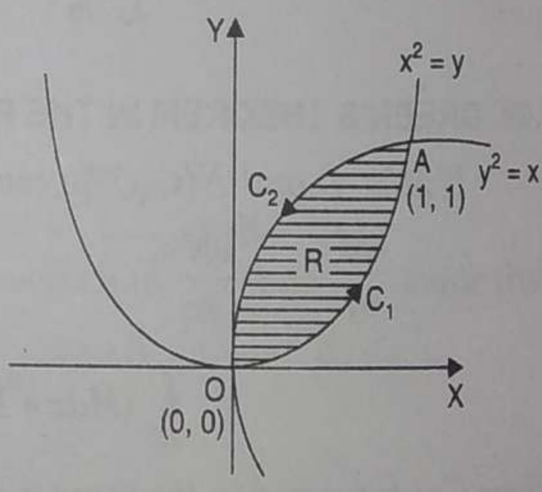
(*a*) $y = \sqrt{x}$, $y = x^2$ (*b*) $x = 0$, $y = 0$, $x + y = 1$.

Sol. (*a*) $y = \sqrt{x}$ *i.e.*, $y^2 = x$ and $y = x^2$ are two parabolas intersecting at O(0, 0) and A(1, 1).

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$



If R is the region bounded by C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 5\left[y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5\left(\frac{3}{10}\right) = \frac{3}{2} \qquad \text{...(1)}$$

Also, $\oint_C (\text{M}dx + \text{N}dy) = \int_{C_1} (\text{M}dx + \text{N}dy) + \int_{C_2} (\text{M}dx + \text{N}dy)$

Along C_1, $x^2 = y$, $\therefore$ $2x\, dx = dy$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along $C_1 = \int_{C_1} (\text{M}dx + \text{N}dy)$

$$= \int_0^1 (3x^2 - 8x^4)\, dx + (4x^2 - 6x \,.\, x^2)\, 2x\, dx = \int_0^1 (3x^2 + 8x^3 - 20x^4)\, dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 = -1$$

Along C_2, $y^2 = x$, $\therefore$ $2y\, dy = dx$ and the limits of y are from 1 to 0.

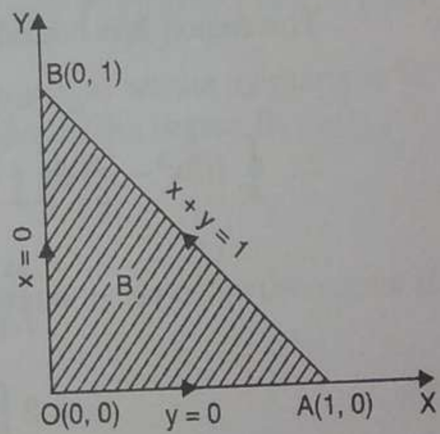
$\therefore$ Line integral along $C_2 = \int_{C_2} (\text{M}dx + \text{N}dy)$

$$= \int_1^0 (3y^4 - 8y^2)\, 2y dy + (4y - 6y^2 \,.\, y)\, dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\, dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$\therefore$ Line integral along $C = -1 + \frac{5}{2} = \frac{3}{2}$ *i.e.*, $\oint_C (\text{M}dx + \text{N}dy) = \frac{3}{2}$ ...(2)

The equality of (1) and (2) verifies Green's theorem in the plane.



(*b*) Here $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \int_0^{1-x} 10y\, dy\, dx$

$$= \int_0^1 5\left[y^2\right]_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2\, dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0 - 1) = \frac{5}{3} \qquad \text{...(1)}$$

Along OA, $y = 0$ $\therefore$ $dy = 0$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along $\text{OA} = \int_0^1 3x^2\, dx = \left[x^3\right]_0^1$

Along AB, $y = 1 - x$ $\therefore$ $dy = -dx$ and the limits of x are from 1 to 0.

$\therefore$ Line integral along AB $= \int_1^0 [3x^2 - 8(1-x)^2]\, dx + [4(1-x) - 6x(1-x)](-dx)$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\, dx = \int_1^0 (-12 + 26x - 11x^2)\, dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$ $\therefore$ $dx = 0$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along BO $= \int_1^0 4y\, dy = \left[2y^2\right]_1^0 = -2$

$\therefore$ Line integral along C (*i.e.*, along OABO) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

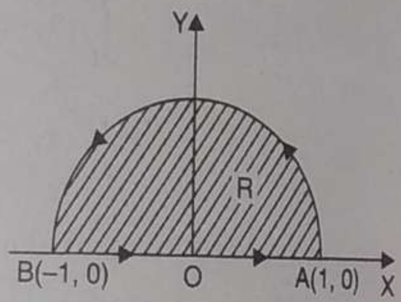
i.e., $$\oint_C (\text{M}dx + \text{N}dy) = \frac{5}{3} \qquad \text{...(2)}$$

The equality of (1) and (2) verifies Green's theorem in the plane.

Example 2. *Use Green's theorem in a plane to evaluate the integral* $\oint_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$ *where C is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle* $x^2 + y^2 = 1$ *in the upper half xy-plane.*

Sol. If R is the region bounded by the closed curve C, then by Green's theorem in the plane, we have

$$\oint_C (\text{M}dx + \text{N}dy) = \iint_R \left(\frac{\partial \text{N}}{\partial x} - \frac{\partial \text{M}}{\partial y}\right) dx\, dy$$



Here $\text{M} = 2x^2 - y^2$, $\text{N} = x^2 + y^2$

$$\frac{\partial \text{M}}{\partial y} = -2y, \quad \frac{\partial \text{N}}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial \text{N}}{\partial x} - \frac{\partial \text{M}}{\partial y} = 2(x + y)$$

The region R is bounded by

$$x = -1,\ x = 1,\ y = 0,\ y = \sqrt{1 - x^2}$$

$$\therefore \oint_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy] = \iint_R 2(x + y)\, dx\, dy$$

$$= \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} 2(x + y)\, dy\, dx = 2\int_{-1}^{1} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= 2\int_{-1}^{1} \left[x\sqrt{1 - x^2} + \frac{1}{2}(1 - x^2)\right] dx$$

$$= \int_{-1}^{1} (1 - x^2)\, dx, \text{ since } \int_{-1}^{1} x\sqrt{1 - x^2}\, dx = 0$$

$$= 2\int_0^1 (1 - x^2)\, dx = 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}.$$

Q. Verify Green's theorem in the plane for

$$\oint_C (xy+y^2)\,dx + x^2\,dy$$

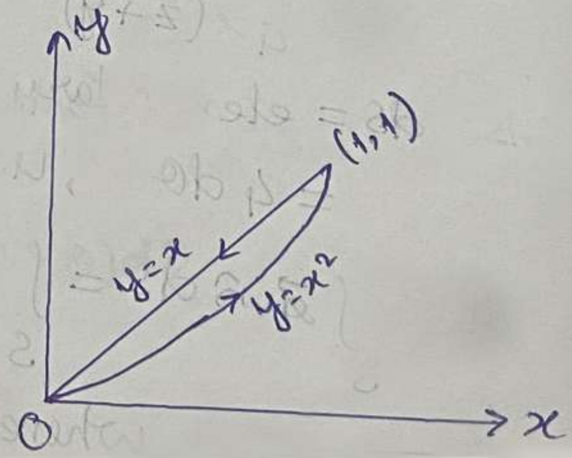
where C is the closed curve of the region bounded by $y=x$ and $y=x^2$

Soln: By Green' theorem in the plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

Here $M = xy+y^2$, $N = x^2$

$$\Rightarrow \frac{\partial M}{\partial y} = x+2y, \quad \frac{\partial N}{\partial x} = 2x$$



The curves $y=x$ and $y=x^2$ intersect at $(0,0)$ and $(1,1)$.
The positive direction in traversing C is a shown in figure.
We have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[xy - x\cdot\frac{y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[x^2 - x^2 - (x^3 - x^4)\right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= \frac{-1}{20}$$

Now, along $y = x^2 \Rightarrow dy = 2x\,dx$

$\therefore$ The line integral $= \int_0^1 \left[\{x \cdot x^2 + (x^2)^2\} dx + x^2(2x\,dx)\right]$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \int_0^1 (x^4 + 3x^3)\,dx$$

$$= \left[\frac{x^5}{5} + \frac{3x^4}{4}\right]_0^1$$

$$= \frac{1}{5} + \frac{3}{4}$$

$$= \frac{4+15}{20}$$

$$= \frac{19}{20}$$

Again, along $y = x \Rightarrow dy = dx$

$\therefore$ The line integral $= \int_1^0 (x \cdot x + x^2)\,dx + x^2\,dx$

$$= \int_1^0 3x^2\,dx$$

$$= \left[x^3\right]_1^0$$

$$= 0 - 1$$

$$= -1$$

$\therefore$ The required line integral $= \frac{19}{20} - 1$

$$= -\frac{1}{20}$$

This verifies Green's theorem. #