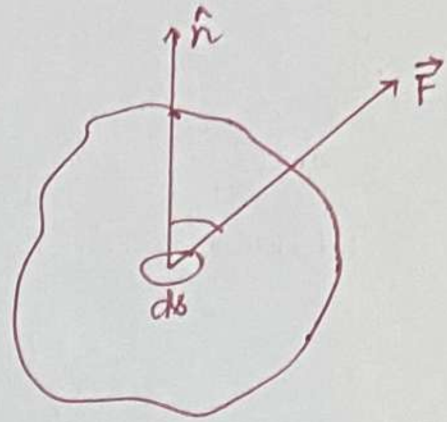
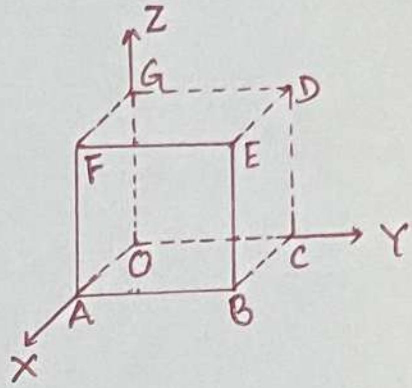
* Surface Integral:

A surface $r = f(u,v)$ is called smooth if $f(u,v)$ posses continuous first order partial derivative.



Let $\vec{F}$ be a vector function and S be the given surface. Surface integral of vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.



$\therefore$ Component of $\vec{F}$ along the normal

$= \vec{F}\cdot\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds and

$$\boxed{\hat{n} = \frac{\nabla f}{|\nabla f|}}, \quad \boxed{ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}} \qquad [\because \text{grad} f = \nabla f]$$

Surface integral of F over $S = \sum \vec{F}\cdot\hat{n} = \boxed{\iint_S (\vec{F}\cdot\hat{n})\,ds}$

* $\boxed{ds = \hat{i}\,dy\,dz + \hat{j}\,dz\,dx + \hat{k}\,dx\,dy}$

Note: (I.) Flux $= \iint_S (\vec{F}\cdot\hat{n})\,ds$, where $\vec{F}$ represents the velocity of a liquid.

(II.) If $\iint_S (\vec{F}\cdot\hat{n})\,ds = 0$, then $\vec{F}$ is said to be a Solenoidal vector point function.

Q. Evaluate $\iint_S (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\,\vec{ds}$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

Soln. Here, $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

& $\phi = x^2 + y^2 + z^2 - a^2$

$\therefore$ Vector normal to the surface $= \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

and $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{|2x\hat{i} + 2y\hat{j} + 2z\hat{k}|} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{|x\hat{i} + y\hat{j} + z\hat{k}|}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \quad [\because x^2 + y^2 + z^2 = a^2]$$

$\therefore$ $\vec{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\left(\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)$

$$= \frac{1}{a}(xyz + xyz + xyz)$$

$$= \frac{3xyz}{a}$$

Now,

$$\iint_S (\vec{F}\cdot\hat{n})\,ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \left[\because |\hat{n}\cdot\hat{k}| = \left|\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)\cdot\hat{k}\right| = \frac{z}{a}\right]$$

$$= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{3xyz}{a}\frac{dx\,dy}{\left(\frac{z}{a}\right)}$$

$$= 3\int_0^a \int_0^{\sqrt{a^2 - x^2}} xy\,dx\,dy = 3\int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2 - x^2}} dx$$

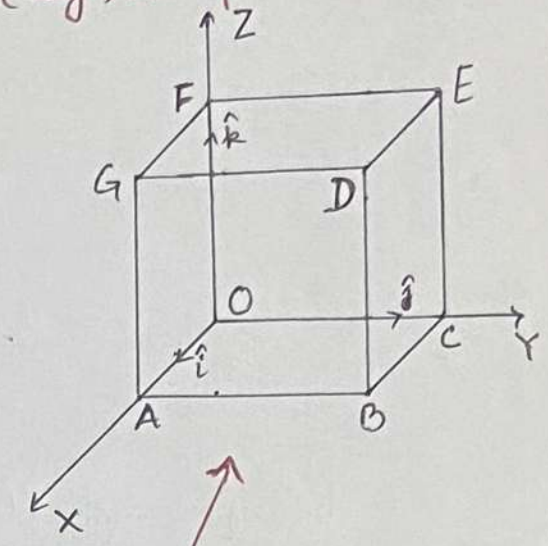
$$= \frac{3}{2}\int_0^a x(a^2 - x^2)\,dx = \frac{3}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{3}{2}\left[a^2\cdot\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\left(a^2\cdot\frac{a^2}{2} - \frac{a^4}{4}\right)$$

$$= \frac{3}{2}\left(\frac{2a^4 - a^4}{4}\right)$$

$$= \frac{3a^4}{8} \quad \#$$

Q. Show that $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes, $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.



Soln. Here, $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAGF} \vec{F}\cdot\hat{n}\,ds + \iint_{BCED} \vec{F}\cdot\hat{n}\,ds + \iint_{ABDG} \vec{F}\cdot\hat{n}\,ds + \iint_{OCEF} \vec{F}\cdot\hat{n}\,ds \longrightarrow (1)$$

Surface	Outward normal	ds	
OABC	$-\hat{k}$	$dxdy$	$z=0$
DEFG	$\hat{k}$	$dxdy$	$z=1$
OAGF	$-\hat{j}$	$dxdz$	$y=0$
BCED	$\hat{j}$	$dxdz$	$y=1$
ABDG	$\hat{i}$	$dydz$	$x=1$
OCEF	$-\hat{i}$	$dydz$	$x=0$

Now,
$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{k})\,dxdy$$
$$= \int_0^1\int_0^1 (-yz)\,dxdy$$
$$= 0 \quad [\because z = 0]$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint_{DEFG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{k}\,dxdy$$
$$= \int_0^1\int_0^1 yz\,dxdy = \int_0^1\int_0^1 y\,dxdy \quad [\because z = 1]$$
$$= \int_0^1 dx\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}\int_0^1 dx\,(1-0)$$
$$= \frac{1}{2}\int_0^1 dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}(1-0)$$
$$= \frac{1}{2}$$

$$\iint_{OAGF} \vec{F}\cdot\hat{n}\,ds = \iint_{OAGF} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 y^2\,dx\,dz = 0 \quad [\because y = 0]$$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = \iint_{BCED} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{j}\,dx\,dz = \iint_{BCED} (-y^2)\,dx\,dz$$

$$= -\int_0^1\int_0^1 1.\,dx\,dz \quad [\because y = 1]$$

$$= -\int_0^1 dx\,[z]_0^1 = -\int_0^1 dx = -[x]_0^1 = -(1-0)$$

$$= -1$$

$$\iint_{OCEF} \vec{F}\cdot\hat{n}\,ds = \iint_{OCEF} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = \iint_{OCEF} (-4xz)\,dy\,dz$$

$$= -4\iint_{OCEF} xz\,dy\,dz$$

$$= 0 \quad [\because x = 0]$$

$$\iint_{ABDG} \vec{F}\cdot\hat{n}\,ds = \iint_{ABDG} (4xz\hat{i} - y\hat{j} + yz\hat{k})\cdot\hat{i}\,dy\,dz = \iint_{ABDG} 4xz\,dy\,dz$$

$$= 4\int_0^1\int_0^1 z\,dy\,dz \quad [\because x = 1]$$

$$= 4\int_0^1 dy\left[\frac{z^2}{2}\right]_0^1 = 2\int_0^1 dy\,(1-0)$$

$$= 2\,[y]_0^1 = 2\,(1-0)$$

$$= 2$$

Putting these values in eqn 1, we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 + \frac{1}{2} + 0 + (-1) + 0 + 2$$

$$= \frac{3}{2}$$ Proved

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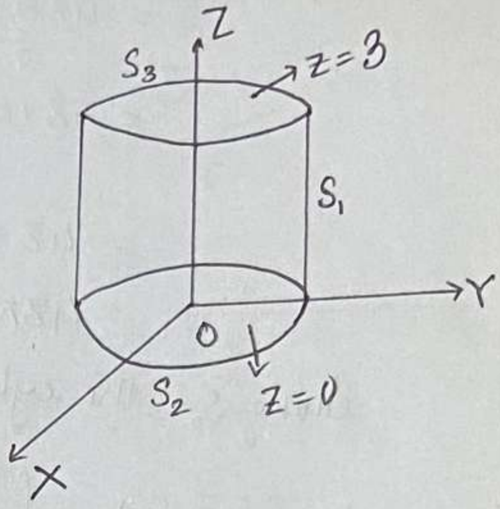
Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2+y^2=4$, $z=0$, $z=3$.

Soln: Given, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$



Let, $\phi = x^2+y^2-4$, then

$$\text{normal vector} = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-4) = 2x\hat{i} + 2y\hat{j}$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}}{|2x\hat{i}+2y\hat{j}|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{2} \quad [\because x^2+y^2=4]$$

Now, the given integral can be written as (along S_1)

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right) ds$$

$$= \frac{1}{2}\iint_{S_1} (4x^2 - 2y^3)\,ds = \iint_{S_1} (2x^2 - y^3)\,ds$$

Since $x^2+y^2 = 4$, so

$x = 2\cos\theta$ and $y = 2\sin\theta$

$\therefore ds = 2\,d\theta\,dz$

[$\because$ Cylindrical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ so that $ds = r\,d\theta\,dz$]

Now,

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \{2(2\cos\theta)^2 - (2\sin\theta)^3\}\,2\,d\theta.dz$$

$$= \int_0^{2\pi}\int_0^3 (16\cos^2\theta - 16\sin^3\theta)\,d\theta\,dz$$

$$= 16\int_0^{2\pi} [z]_0^3 (\cos^2\theta - \sin^3\theta)\,d\theta$$

$$= 16\times 3\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta$$

$$= 48 \int_0^{2\pi} \cos^2\theta \, d\theta - 48 \int_0^{2\pi} \sin^3\theta \, d\theta$$

$$= 48 \cdot \int_0^{2\pi} \cos^2 d\theta - 48 \cdot (0)$$

$$= 48 \times 4 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$\boxed{\int_0^{\pi/2} \underset{\underset{\cos^n x\,dx}{||}}{\sin^n x\,dx} = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \end{cases}}$$

$$= 48 \times 4 \times \frac{1}{2} \times \frac{\pi}{2} \quad [\because \text{By Walli's formula}]$$

$$= 48\pi$$

Along $S_2 =$ the cylinder base in the plane $z = 0$, then $z = 0$, $\hat{n} = -\hat{k}$

$$\therefore \iint_{S_2} \vec{F} \cdot \hat{n} \, ds = \iint_{S_2} (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot (-\hat{k}) \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint_{S_2} -z^2 \, dx\,dy$$

$$= 0$$

Along $S_3 =$ the cylinder top in the plane $z = 3$, then $z = 3$, $\hat{n} = \hat{k}$

$$\therefore \iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \iint_{S_3} (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \hat{k} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint_{S_3} z^2 \, dx\,dy = 9 \iint_{S_3} dx\,dy$$

$$= 9 \times \pi (2)^2$$

$$= 36\pi$$

$\boxed{\because \text{Surface area of } x^2 + y^2 = r^2 \text{ is } \pi r^2}$

[$\because$ surface area of $x^2 + y^2 = 4$ is 4π]

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} \vec{F} \cdot \hat{n} \, ds + \iint_{S_2} \vec{F} \cdot \hat{n} \, ds + \iint_{S_3} \vec{F} \cdot \hat{n} \, ds$$

$$= 48\pi + 0 + 36\pi$$

$$= 84\pi$$

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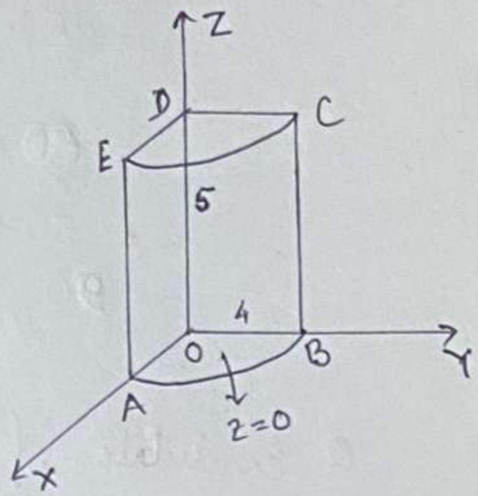
Q. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$



Soln. Given, $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$

Let $f = x^2+y^2-16$, then

normal vector $= \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-16)$$

$$= 2x\hat{i} + 2y\hat{j}$$

& $$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}+2y\hat{j}}{|2x\hat{i}+2y\hat{j}|} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{x\hat{i}+y\hat{j}}{4} \quad [\because x^2+y^2=16]$$

The projection of the surface on yz-plane, then

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dy\,dz}{\left|\left(\frac{x\hat{i}+y\hat{j}}{4}\right)\cdot\hat{i}\right|} = \frac{4}{x}\,dy\,dz$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_S (z\hat{i}+x\hat{j}-3y^2z\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{\cancel{4}}\right)\frac{\cancel{4}}{x}\,dy\,dz$$

$$= \iint_S (xz+xy)\frac{dy\,dz}{x}$$

$$= \iint_S (z+y)\,dy\,dz$$

Since, z limits from 0 to 5

$x^2+y^2=16 \Rightarrow$ yz-plane, $x=0$

$y^2=16 \Rightarrow y=\pm 4$

In the first octant, $y \to 0$ to 4

$$\therefore \iint_S \vec{A}\cdot\hat{n}\,ds = \int_{y=0}^{4}\int_{z=0}^{5}(z+y)\,dy\,dz$$

$$= \int_{y=0}^{4}\left[\frac{z^2}{2}+yz\right]_0^5 dy$$

$$= \int_0^4 \left(\frac{25}{2} + 5y\right) dy$$

$$= \frac{25}{2}[y]_0^4 + 5\left[\frac{y^2}{2}\right]_0^4$$

$$= \frac{25}{2} \times (4-0) + 5\left(\frac{16}{2} - 0\right)$$

$$= 50 + 40$$

$$= 90$$

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Q. Evaluate $\iint_S \vec{A} \cdot \hat{n}\, ds$, where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Soln. Given, $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$

Let $f = 2x + y + 2z - 6$, then

normal vector $= \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z - 6)$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{|2\hat{i} + \hat{j} + 2\hat{k}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}$$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

and $ds = \dfrac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \dfrac{dy\,dz}{|\hat{n} \cdot \hat{i}|} = \dfrac{dx\,dz}{|\hat{n} \cdot \hat{j}|}$

Now, $$\iint_S \vec{A} \cdot \hat{n}\, ds = \iint_S \{(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\} \cdot \left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right) \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \frac{1}{3} \iint_S \left[2(x+y^2) - 2x + 4yz\right] \frac{dx\,dy}{\frac{2}{3}}$$

$$\left[\because |\hat{n} \cdot \hat{k}| = \left|\left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right) \cdot \hat{k}\right| = \frac{2}{3}\right.$$

$$= \frac{\cancel{2}}{\cancel{3}} \iint_S (\cancel{x} + y^2 - \cancel{x} + 2yz) \frac{\cancel{3}}{\cancel{2}}\, dx\,dy$$

$$\iint_S (y^2+2yz)\,dx\,dy \longrightarrow ①$$

Since $2x+y+2z=6 \Rightarrow 2x+y=6$ [on xy-plane, $z=0$]

$$\Rightarrow y = 6-2x$$

putting $y=0 \Rightarrow 2x=6 \Rightarrow x=3$

From eqn ① $\Rightarrow$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_{x=0}^{3}\int_{y=0}^{6-2x} \{y^2+y(6-2x-y)\}\,dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} (y^2+6y-2xy-y^2)\,dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} (6y-2xy)\,dx\,dy$$

$$= \int_0^3 dx\left[\cancel{6}^{3}\cdot\frac{y^2}{2} - 2x\frac{y^2}{2}\right]_0^{6-2x} = \int_0^3 dx\left[3y^2 - xy^2\right]_0^{6-2x}$$

$$= \int_0^3 dx\left[(3-x)(6-2x)^2\right] = \int_0^3 (3-x)(36-24x+4x^2)\,dx$$

$$= \int_0^3 (108-72x+12x^2-36x+24x^2-4x^3)\,dx$$

$$= \int_0^3 (108-108x+36x^2-4x^3)\,dx$$

$$= \left[108x - 108\frac{x^2}{2} + 36\frac{x^3}{3} - 4\frac{x^4}{4}\right]_0^3$$

$$= \left[108x - 54x^2 + 12x^3 - x^4\right]$$

$$= 108\times 3 - 54\times 9 + 12\times 27 - 81$$

$$= 324 - 486 + 324 - 81$$

$$= 81 \#$$

Example *Evaluate* $\iint_S \vec{A}.\hat{n}\,d\,S$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\,\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

Solution. Here, $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

Given surface $f(x, y, z) = 2x + 3y + 6z - 12$

Normal vector $= \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\hat{n}$ = unit normal vector at any point (x, y, z) of $2x + 3y + 6z = 12$

$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

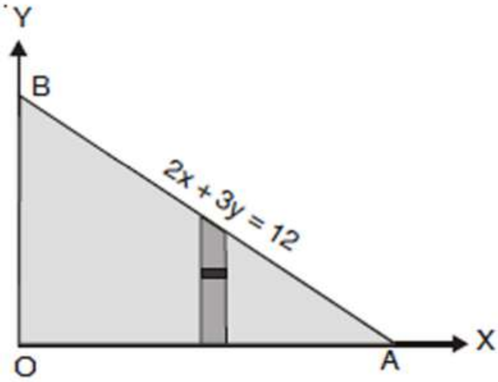
$$dS = \frac{dx\,dy}{\hat{n}.\hat{k}} = \frac{dx\,dy}{\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}).\hat{k}} = \frac{dxdy}{\frac{6}{7}} = \frac{7}{6}\,dx\,dy$$

Now, $\iint \vec{A}.\hat{n}\,dS = \iint (18z\hat{i} - 12\hat{j} + 3y\hat{k}).\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\frac{7}{6}\,dx\,dy$

$$= \iint (36z - 36 + 18y)\frac{dx\,dy}{6} \qquad = \iint (6z - 6 + 3y)\,dx\,dy$$

Putting the value of $6z = 12 - 2x - 3y$, we get

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$



$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6 - 2x)\,dx\,dy$$

$$= \int_0^6 (6 - 2x)\,dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6 - 2x)\,dx\,(y)_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6 - 2x)\frac{1}{3}(12 - 2x)\,dx = \frac{1}{3}\int_0^6 (4x^2 - 36x + 72)\,dx$$

$$= \frac{1}{3}\left[\frac{4x^3}{3} - 18x^2 + 72x\right]_0^6 = \frac{1}{3}[4 \times 36 \times 2 - 18 \times 36 + 72 \times 6] = \frac{72}{3}[4 - 9 + 6] = 24$$ **Ans.**

Q: Find the value of the surface integral $\iint_S (2x^2y\,dy\,dz - y^2dz\,dx + 4xz^2dx\,dy)$ where S is the curved surface of the cylinder $y^2+z^2=9$, bounded by the planes $x=0, x=2$.

Soln. We have,

$\hat{n}\,ds = \hat{i}\,dydz + \hat{j}\,dzdx + \hat{k}\,dxdy$ in terms of the projection of $\vec{ds}$ on the co-ordinate planes.

Let, $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$

and $\phi = y^2+z^2-9$, then normal vector

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y^2+z^2-9) = 0.\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2y\hat{j}+2z\hat{k}}{|2y\hat{j}+2z\hat{k}|} = \frac{y\hat{j}+z\hat{k}}{\sqrt{y^2+z^2}} = \frac{y\hat{i}+z\hat{k}}{3} \quad [\because y^2+z^2=9]$$

Now, the given integral can be written as

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot\left(\frac{y\hat{j}+z\hat{k}}{3}\right)ds$$

$$= \frac{1}{3}\iint_S (-y^3 + 4xz^3)\,ds$$

Since $y^2+z^2=9$, so $\quad$ [$\because$ Cylindrical polar coordinates $y=r\sin\theta$ & $z=r\cos\theta$ so that $ds = r\,d\theta\,dx$]

$y = 3\sin\theta$ & $z = 3\cos\theta$

$\therefore ds = 3\,d\theta\,dx$

Now, $$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{1}{\cancel{3}}\int_{\theta=0}^{2\pi}\int_{x=0}^{2}\left\{-(3\sin\theta)^3 + 4x(3\cos\theta)^3\right\}\cancel{3}\,d\theta\,dx$$

$$= 27\int_{\theta=0}^{2\pi}\int_{x=0}^{2}(-\sin^3\theta + 4x\cos^3\theta)\,d\theta\,dx$$

$$= 27\int_0^{2\pi}\left[-\sin^3\theta\int_0^2 dx + 4\cos^3\theta\int_0^2 x\,dx\right]d\theta$$

$$= 27\int_0^{2\pi}\left[-x\sin^3\theta + 4\times\frac{x^2}{2}\cos^3\theta\right]_0^2 d\theta$$

$$= 27\int_0^{2\pi}\left(-2\sin^3\theta + 8\cos^3\theta\right)d\theta$$

$$= -54\int_0^{2\pi}\sin^3\theta\, d\theta + 216\int_0^{2\pi}\cos^3\theta\, d\theta$$

$$= 0 + 216\times 2\int_0^{\pi}\cos^3\theta\, d\theta$$

$$= 0 + 432(0)$$

$$= 0$$

#

[$\because$ By property of definite integral, we have

$$\int_0^{2\pi}\sin^3\theta\, d\theta = 0$$

& $\displaystyle\int_0^{2\pi}\cos^3\theta\, d\theta = 2\int_0^{\pi}\cos^3\theta\, d\theta = 0$]

1. Evaluate $\iint_S \vec{A}.\hat{n}\, ds$, where $\vec{A} = (x+y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant. **Ans.** 81

2. Evaluate $\iint_S \vec{A}.\hat{n}\, ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. **Ans.** 90

3. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\,\hat{k}$ and $\vec{S} = 2t^2\hat{i} + 6t\hat{k}$, evaluate $\int_0^2 \vec{r}\cdot\vec{S}\, dt$. **Ans.** 12

4. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, dS$, where, $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant. **Ans.** 24

5. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where, $F = 2yx\hat{i} - yz\hat{j} + x^2\hat{k}$ over the surface S of the cube bounded by the coordinate planes and planes $x = a$, $y = a$ and $z = a$. **Ans.** $\frac{1}{2}a^4$

6. If $\vec{F} = 2y\hat{i} - 3\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, and $z = 6$, then evaluate $\iint_S \vec{F}\cdot\hat{n}\, dS$. **Ans.** 132